A Companion to

Advanced Problems

in

Core Mathematics

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Introduction

This companion booklet contains answers together with comments to selected questions from Dr Stephen Siklos' 'Advanced Problems in Core Mathematics' booklet.[1] Why go to the the trouble of compiling another set of answers and comments, you might ask, when the original booklet already contains excellent ones? Well, partly for fun. I have taught from 'Advanced Problems in Core Mathematics' three times now and have enjoyed it more each time. And, since my answers are sufficiently different from the model answers a good proportion of the time, typing them up seemed like a worthwhile undertaking. I also think that a slightly different format, specifically one giving each answer in full and then followed by comments, might suit some students a little better.

There is also another motivation for compiling this booklet, and that is my own experience in sitting the Further Mathematics A and B STEP examinations. Preparing for them, which I did largely on my own, was a thankless task, and when I came to sit the examinations, I was petrified. Cambridge was far from easy when I got there, but I think the loss of my confidence as a mathematician, confidence that has taken many years to recover, started with STEP. So when I had the opportunity to teach STEP level mathematics, I was unsure to say the least! Fortunately I found that being more than twenty years older meant that I could manage most of the questions without too much difficulty. Furthermore, it became obvious as I worked through the questions that it was never the intention of the examiners, Dr Siklos being chief among them, not to help the students, rather the complete opposite. You will see in what follows that I point out many of the occasions where the examiners try to help you as much as they can, in fact.

To continue in this vein for a moment, because of my own past experience I have tried not to give the impression that every one of answers flowed easily for me this time around, since this would not help your confidence. Mathematicians would have you believe that it is all plain sailing, but I can assure you that everyone struggles sometimes. What I have done therefore is to type up the answers pretty much as I wrote them at the time and in subsequent revisions, though the temptation has been strong, I have polished the comments whilst leaving the answers as-is.

Now for some advice. Do not look at the answers beforehand. A little struggle is necessary, it toughens you up, but on the other hand do not overdo it as I did. I would also advise not looking at the comments that appear right under the questions in Dr Siklos' booklet, for two reasons. Firstly, if they are helpful, in reading them you have lost the chance of attempting the questions blind, and you must do this. Secondly, a lot of the time they will not be helpful. They often contain anecdotal or tangential information that is fascinating and sometimes humorous, but it will frustrate you if you are looking for a leg up. So I would recommend that you place a book or something over the comment immediately you turn the page on to a new question, and consider comments a digestive rather than an aperitif.

I should also mention that I have only given answers to the pure mathematics questions. I could do most of the others but I do not have the same level of engagement with probability, statistics or mechanics and so I felt that my answers and advice would likely seem hollow. All I can do in these circumstances is to commend the model answers to you.

Good luck with it!

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[1] I was gratified on googling it to find that the sibling 'Advanced Problems in Mathematics' booklet is still freely available from the STEP section of the Cambridge University admissions website. This booklet is a companion to the 'Advanced Problems in Core Mathematics' booklet, however, which used to be on the Cambridge University website but now no longer appears to be. If you cannot find it online, therefore, please feel free to email me for one.

Question 1

We split each of the modulus functions into branches:

$$|x+1| = \begin{cases} x+1 & x \geqslant -1 \\ -(x+1) & x \leqslant -1 \end{cases} \qquad |x| = \begin{cases} x & x \geqslant 0 \\ -x & x \leqslant 0 \end{cases}$$

$$3|x-1| = \begin{cases} 3(x-1) & x \geqslant 1 \\ -3(x-1) & x \leqslant 1 \end{cases} \qquad 2|x-2| = \begin{cases} 2(x-2) & x \geqslant 2 \\ -2(x-2) & x \leqslant 2 \end{cases}$$

From this we can see that there are five intervals to consider:

$$\begin{aligned} 2 &\leqslant x \\ 1 &\leqslant x \leqslant 2 \\ 0 &\leqslant x \leqslant 1 \\ -1 &\leqslant x \leqslant 0 \\ & x \leqslant -1 \end{aligned}$$

$2 \leqslant x$:

$$\begin{aligned} (x+1) - x + 3(x-1) - 2(x-2) &= x+2 \\ x + 1 - x + 3x - 3 - 2x + 4 &= x+2 \\ 2 &= 2 \end{aligned}$$

So the equation holds for any x.

$1 \leqslant x \leqslant 2$:

$$\begin{aligned} (x+1) - x + 3(x-1) + 2(x-2) &= x+2 \\ x + 1 - x + 3x - 3 + 2x - 4 &= x+2 \\ 5x - 6 &= x+2 \\ 4x &= 8 \\ x &= 2 \end{aligned}$$

$0 \leqslant x \leqslant 1$:

$$\begin{aligned} (x+1) - x - 3(x-1) + 2(x-2) &= x+2 \\ x + 1 - x - 3x + 3 + 2x - 4 &= x+2 \\ -x &= x+2 \\ x &= -1 \end{aligned}$$

$-1 \leqslant x \leqslant 0$:

$$\begin{aligned} (x+1) + x - 3(x-1) + 2(x-2) &= x+2 \\ x + 1 + x - 3x + 3 + 2x - 4 &= x+2 \\ x &= x+2 \\ 0 &= 2 \end{aligned}$$

$x \leqslant -1$:

$$\begin{aligned}
-(x+1) + x - 3(x-1) + 2(x-2) &= x + 2 \\
-x - 1 + x - 3x + 3 + 2x - 4 &= x + 2 \\
-x - 2 &= x + 2 \\
0 &= 2x + 4 \\
0 &= x + 2 \\
x &= -2
\end{aligned}$$

To summarise:

1. For $2 \leqslant x$ the equation holds for any x. Therefore $x \leqslant 2$ is a solution of the main equation.
2. For $1 \leqslant x \leqslant 2$ the equation holds for $x = 2$, adding no more solutions.
3. For $0 \leqslant x \leqslant 1$ the equation only holds for $x = -1$, again adding no more solutions.
4. For $-1 \leqslant x \leqslant 0$ the equation is unsolvable, so again no more solutions.
5. For $x \leqslant -1$ the equation holds for $x = -2$, adding another solution to the main equation.

Therefore the main equation holds for $x = -2$ and $x \geqslant 2$.

Comments

The only thing you can ever do with equations involving modulus functions is to split the modulus functions down into their constituent branches, solve the resulting equations and then gather the results. This question requires no more than the careful application of this technique and some basic algebra.

Note how I lay out the calculations in full, even though they are straightforward. It costs very little to do this time-wise and it means that if you make a mistake, you stand a good chance of finding it. Here it is not easy to detect mistakes because the answers are not given and you therefore have nothing to aim at, but it is good to get into the practice of being punctilious.

You may ask, what does it mean to solve an equation where the variables are constrained and the solutions turn out to lie outside of those constraints? Or what if, as in the fourth case, working the equation through results in nonsense? Let me go through the cases. In the first case, we get the seemingly unhelpful $2 = 2$, and yet I concluded from this that the equation holds for any x. If I were to solve the equation $x + 2 = x + 2$, however, it would also boil down to $2 = 2$ and you can clearly see that $x+2 = x+2$ holds for any x. Put another way, if we can eliminate a variable entirely from an equation, then it holds for any value of that variable. In the second case, the answer lies inside of the constraint and so there is hopefully nothing to think about. In the third case we get an answer that lies outside of the constraints. The best explanation I can give for this is that when we solved the equation we effectively solved it for all x, the information about the branches was temporarily discarded, and so it is perhaps not surprising that a solution should pop up outside of the range we were considering. In the fourth case, which is probably the most alarming, we get $0 = 2$. This is known as a contradiction. It comes about because the derivation began with a supposition that was wrong, that is, we equated two expressions that could never be equal. To see the point, consider the equation $x + 1 = x + 2$, which also leads to a contradiction. Here the supposition is obviously wrong to start with. Sometimes it is not at all obvious that our suppositions are wrong, however, and we just have to see where they lead. But if they lead to a contradiction then we need not worry overly, just disregard the supposition. The fifth and last case should be fine.

This may be the first time that you have come across the technique of splitting modulus functions into their branches but you will doubtless come across it again, so try to make an effort remember it. At this level you do not have the luxury of forgetting.

Question 2

(i) We calculate just the terms involving x^6:

$$(1 - 2x + 3x^2 - 4x^3 + 5x^4)^3$$

$$= (1 - 2x + 3x^2 - 4x^3 + 5x^4)$$
$$\times (1 - 2x + 3x^2 - 4x^3 + 5x^4)$$
$$\times (1 - 2x + 3x^2 - 4x^3 + 5x^4)$$

We look at all the permutations of products of terms where the coefficients of x add up to 6. For example $0+2+4=6$, and we can multiply the terms $1 = x^0$, $3x^2$ and $5x^4$ together in six different ways. In total we have:

$$6 \times 1.3x^2.5x^4$$
$$3 \times 1. -4x^3. -4x^3$$
$$3 \times -2x. -2x.5x^4$$
$$6 \times -2x.3x^2. -4x^3$$
$$1 \times 3x^2.3x^2.3x^2$$

Summing we have $90 + 48 + 60 + 144 + 27 = 369$.

(ii) We recall that the binomial expansion works for negative powers, too:

$$(1+x)^{-n} = 1 + \frac{(-n)x}{1!} + \frac{(-n)(-n-1)x^2}{2!} + \cdots$$

This gives:

$$(1+x)^{-2} = 1 + \frac{(-2)x}{1!} + \frac{(-2)(-3)x^2}{2!} + \frac{(-2)(-3)(-4)x^3}{3!} + \cdots = 1 - 2x + 3x^2 - 4x^3 + \cdots$$

$$(1+x)^{-6} = 1 + \frac{(-6)x}{1!} + \frac{(-6)(-7)x^2}{2!} + \frac{(-6)(-7)(-8)x^3}{3!} + \cdots = 1 - 6x + 21x^2 - 56x^3 + \cdots$$

Writing out the first expansion more fully...

$$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - 6x^5 + 7x^6 - x^7[\cdots]$$

..we note that the first eight terms constitute the factor in the given expression. Then we note:

$$[(1+x)^{-2}]^3 = (1+x)^{-6}$$

We can equate each side of the equation term by term and since terms of order 7 or more cannot contribute to the x^6 term on the left, its coefficient must be the same as the coefficient of the x^6 term on the right. This we can calculate directly:

$$\frac{(-6)(-7)(-8)(-9)(-10)(-11)}{6!} = \frac{6 \times 7 \times 8 \times 9 \times 10 \times 11}{6 \times 5 \times 4 \times 3 \times 2 \times 1} = 7 \times 2 \times 3 \times 11 = 42 \times 11 = 462$$

Comments

The trick with part (i) is to realise that you must not waste your time writing out the whole expansion. You may be wondering, therefore, why I went to the time consuming lengths of essentially writing out the expression three times. I did this merely to make the calculation easier, because I could then mark out the "routes" between the terms that give rise to all the permutations. I also made a point of giving a reasonable explanation of what I was up to. I did not do the multiplications and additions to arrive at the coefficient in my head, either, I wrote them out in full. I cannot emphasise enough what a bad idea I think it is to attempt mental arithmetic under the pressure of an examination. It is likely to be no quicker, firstly, and secondly, it is almost guaranteed to result in a mistake. Additionally, you have no way of checking your work and lastly, the examiner cannot at least give you some marks for your working out.

For part (ii), it may not be at all obvious how considering the binomial expansions of $(1+x)^{-2}$ and $(1+x)^{-6}$ enables you to find the coefficient of x^6 in the given expression. The only thing you can do in these situations is to take the advice on offer and dutifully write out the expansions, in the hope that in doing so the penny drops. And when you do so, voila! The first seven terms in the expansion for $(1+x)^{-2}$ are precisely the terms in the given expression. Furthermore, it should have been clear from the first few terms where the expansion was going, so hopefully you did not have to slavishly work them all out. This is the first of many occasions where it is evident that the examiner is trying to help you. If you did not get the trick of equating $(1+x)^{-2}$ to $(1+x)^{-6}$, which leads to the realisation that all you need to do is calculate the coefficient of x^6 in the latter expression, by the way, then you could still have employed the technique used in part (i).

There is much I could write on the binomial expansion, but I will leave off and simply recommend that you remember the form given here. I seldom recommend trying to remember things parrot fashion, I think the best way to remember something in mathematics is to understand it. In this case, however, I think it is fine just to make a mental note. Finally, it is not entirely true to say that the binomial expansion "works" for negative powers, since it might not converge. You would not be penalised for not knowing this fact at this level, however, so on reflection I left this erroneous comment in. I also left in the weird 1.3, meaning 1×3, because again, the examiner would doubtless know what I was driving at.

Question 3

10
5 5
5 2 2 1
5 2 1 1 1
5 1 1 1 1 1
2 2 2 2 2
2 2 2 2 1 1
2 2 2 1 1 1 1
2 2 1 1 1 1 1 1
2 1 1 1 1 1 1 1 1
1 1 1 1 1 1 1 1 1 1

There are eleven ways in total.

$$\begin{array}{ll} 20 & \text{1 way} \end{array}$$

$$\left.\begin{array}{ll} 10\ 10 & \\ & 5\ 5 \\ & 5\ 2\ 2\ 1 \\ & \vdots \\ & 5\ 1\ 1\ 1\ 1\ 1 \end{array}\right\}\text{11 ways}$$

$$\left.\begin{array}{ll} 5\ 5\ 5\ 5 & \\ & 2\ 2\ 1 \\ \vdots & \\ & 1\ 1\ 1\ 1\ 1 \end{array}\right\}\text{4 ways}$$

$$\left.\begin{array}{ll} 5\ 5 & 2\ 2\ 2\ 2\ 2 \\ & 2\ 2\ 2\ 2\ 1\ 1 \\ \vdots & \\ & 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1 \end{array}\right\}\text{6 ways}$$

$$\left.\begin{array}{ll} 5 & 2\ 2\ 2\ 2\ 2\ 2\ 2\ 1 \\ & 2\ 2\ 2\ 2\ 2\ 2\ 1\ 1\ 1 \\ \vdots & \\ & 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1 \end{array}\right\}\text{8 ways}$$

$$\left.\begin{array}{l} 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2 \\ \vdots \\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1 \end{array}\right\}\text{11 ways}$$

There are $1 + 11 + 4 + 6 + 8 + 11 = 41$ ways in total.

Comments

I did not see the point of this question when I came across it and did not enjoy typing it up much just now either! It is the kind of question that anyone with basic numeracy skills can attempt, but I reckon few will get entirely correct the first time, that is if they are completely honest about it. It is an exercise in being systematic and perhaps I should have done better. The only useful advice I think I can give is that you should be as clear as possible about your reasoning in these kinds of situations. This gives the examiner the best possible chance of awarding you some marks even if, as teachers like to say, you got the question not quite right.

Question 4

$$-\left.\begin{aligned} 2yz + zx - 5xy &= 2 \\ yz - zx + 2xy &= 1 \\ yz - 2zx + 6xy &= 3 \end{aligned}\right\} \qquad \left.\begin{aligned} 2yz + zx - 5xy &= 2 \\ \times 2 \quad yz - zx + 2xy &= 1 \\ zx - 4xy &= -2 \end{aligned}\right\}$$

$$-\left.\begin{aligned} 2yz + zx - 5xy &= 2 \\ 2yz - 2zx + 4xy &= 1 \\ zx - 4xy &= -2 \end{aligned}\right\} \qquad \left.\begin{aligned} 2yz + zx - 5xy &= 2 \\ \div 3 \quad 3zx - 9xy &= 0 \\ zx - 4xy &= -2 \end{aligned}\right\}$$

$$-\left.\begin{aligned} 2yz + zx - 5xy &= 2 \\ zx - 3xy &= 0 \\ zx - 4xy &= -2 \end{aligned}\right\} \qquad \left.\begin{aligned} 2yz + zx - 5xy &= 2 \\ zx - 3xy &= 0 \\ xy &= 2 \end{aligned}\right\}$$

This gives $xy = 2$ directly, then:

$$\begin{aligned} zx - 3 \times 2 &= 0 \\ zx - 6 &= 0 \\ zx &= 6 \\ 2yz + 6 - 5 \times 2 &= 2 \\ 2yz - 4 &= 2 \\ 2yz &= 6 \\ yz &= 3 \end{aligned}$$

Lastly:

$$\begin{aligned} xy \times yz \times zx &= 2 \times 3 \times 6 \\ (xyz)^2 &= 36 \\ xyz &= \pm 6 \end{aligned}$$

To find x, y and z we substitute:

$$xy = 2: \quad 2z = \pm 6 \qquad zx = 6: \quad 6y = \pm 6 \qquad yz = 3: \quad 3x = \pm 6$$
$$z = \pm 3 \qquad\qquad y = \pm 1 \qquad\qquad x = \pm 2$$

If we pick $x = +2$, then since $xy = 2$ we must have $y = +1$ and since $yz = 3$, we must have $z = +3$. Similarly, if we pick $x = -2$ then, by $xy = 2$ again, we must have $y = -1$ and by $yz = 3$ again, we must have $z = -3$. Therefore the two sets of solutions are:

$$x = +2, y = +1, z = +3$$

$$x = -2, y = -1, z = -3$$

Comments

I have mentioned already that you should never miss out steps in your calculations, regardless of how trivial they seem. Solving simultaneous equations is a great example. You will doubtless make the odd mistake. If you write out each of the steps carefully, however, you will be able to find these mistakes, even under examination conditions. In this vein, you will notice that at each step only one of the simultaneous equations changes. The difference between the first and second sets of equations is only the third equation, for example. Also note that I have shown the operation that results in the change. Even if you just multiply a row by a constant before adding or subtracting, do so separately, as in the second step.

Not all of the permutations of x, y and z will work and consequently a little care is needed at the end, but I expect you are getting used to that by now. Oh, and I like to group simultaneous equations with curly brackets, as you can see, but you do not have to.

Question 5

$$x_1 = \frac{2}{3}: \qquad \begin{aligned} 3 &= x_1 + \frac{2}{x_2} \\ &= \frac{2}{3} + \frac{2}{x_2} \\ -\frac{2}{x_2} &= \frac{2}{3} - 3 \\ &= \frac{2}{3} - \frac{9}{3} \end{aligned} \qquad \begin{aligned} -\frac{2}{x_2} &= -\frac{7}{3} \\ \frac{x_2}{2} &= \frac{3}{7} \\ \therefore x_2 &= \frac{6}{7} \end{aligned}$$

$$x_2 = \frac{6}{7}: \qquad \begin{aligned} 3 &= x_2 + \frac{2}{x_3} \\ &= \frac{6}{7} + \frac{2}{x_3} \\ \frac{2}{x_3} &= 3 - \frac{6}{7} \\ &= \frac{21}{7} - \frac{6}{7} \end{aligned} \qquad \begin{aligned} \frac{2}{x_3} &= \frac{15}{7} \\ \frac{x_3}{2} &= \frac{7}{15} \\ \therefore x_3 &= \frac{14}{15} \end{aligned}$$

$$x_3 = \frac{14}{15}: \qquad \begin{aligned} 3 &= x_3 + \frac{2}{x_4} \\ &= \frac{14}{15} + \frac{2}{x_4} \\ \frac{2}{x_4} &= 3 - \frac{14}{15} \\ &= \frac{45}{15} - \frac{14}{15} \end{aligned} \qquad \begin{aligned} \frac{2}{x_4} &= \frac{31}{15} \\ \frac{x_4}{2} &= \frac{15}{31} \\ \therefore x_4 &= \frac{30}{31} \end{aligned}$$

So the sequence is:

$$x_1 = \frac{2}{3}, x_2 = \frac{6}{7}, x_3 = \frac{14}{15}, x_4 = \frac{30}{31}, x_5 = \frac{62}{63}, \cdots$$

This gives:

$$x_n = \frac{2^{n+1} - 2}{2^{n+1} - 1}$$

Writing $P(n)$ for $x_n = \dfrac{2^{n+1} - 2}{2^{n+1} - 1}$, we need to prove...

1. $P(1)$
2. $P(k) \Rightarrow P(k+1)$ for all $k \geqslant 1$

...in which case we can assert, by induction, that $P(n)$ holds for all $n \geqslant 1$.

In order to prove $P(1)$, we need to show that the expression for x_1 equates to $\frac{2}{3}$:

$$\left.\frac{2^{n+1}-2}{2^{n+1}-1}\right|_{n=1} = \frac{2^{1+1}-2}{2^{1+1}-1}$$
$$= \frac{2^1-2}{2^2-1}$$
$$= \frac{2}{3}$$

In order to prove that $P(k) \Rightarrow P(k+1)$, we suppose that $P(k)$ holds, that is:

$$x_k = \frac{2^{k+1}-2}{2^{k+1}-1}$$

Then we can make use of the equation linking x_k to x_{k+1}:

$$\begin{aligned}
3 &= x_k + \frac{2}{x_{k+1}} \\
&= \frac{2(2^k-1)}{2^{k+1}-1} + \frac{2}{x_{k+1}} \\
&= \frac{2x_{k+1}(2^k-1)+2(2^{k+1}-1)}{x_{k+1}(2^{k+1}-1)} \\
3x_{k+1}(2^{k+1}-1) &= 2x_{k+1}(2^k-1)+2(2^{k+1}-1) \\
3x_{k+1}(2^{k+1}-1)-2x_{k+1}(2^k-1) &= 2(2^{k+1}-1) \\
x_{k+1}(3(2^{k+1}-1)-2(2^k-1)) &= 2(2^{k+1}-1) \\
x_{k+1}(3.2^{k+1}-3-2^{k+1}+2) &= 2(2^{k+1}-1) \\
x_{k+1}(2.2^{k+1}-1) &= 2(2^{k+1}-1) \\
x_{k+1}(2^{k+2}-1) &= 2^{k+2}-2 \\
x_{k+1} &= \frac{2^{k+2}-2}{2^{k+2}-1}
\end{aligned}$$

This last equation is $P(k+1)$ and so we are done.

Comments

This question ups the ante and the answer may have come as something of a shock. Firstly, I hope at least that the first part of was doable. You are only asked to make an educated guess and so it is fine just to work the first few terms out. I did the first four, just to be sure.

As for the second part, I find the problem with induction at this level is that students never appreciate what exactly they are required to prove. They usually end up effectively trying to prove the Principle of Induction by example and since they usually do not understand what it is to start with, they get nowhere. Let me start, therefore, by stating it reasonably formally:

$$\left.\begin{array}{r} P(0) \\ \forall k\,(P(k) \Rightarrow P(k+1)) \end{array}\right\} \Rightarrow \forall n P(n)$$

$P(0)$ is called the base case and $P(k) \Rightarrow P(k+1)$ is the induction step. In the question we start from 1 not 0 but the principle is the same, if you'll pardon the pun. Moving swiftly on, here it is in plain English:

"If we can prove both that $P(0)$ holds and that $P(k) \Rightarrow P(k+1)$ holds for all k, then we are free to assert that $P(n)$ holds for all n."

$P(n)$ is a proposition indexed by the natural number n, or you might prefer to think of it as a collection of separate propositions $P(0)$, $P(1)$, etc. We can talk of a proposition as holding in the sense that it can be asserted or, if you like, that it is "true". It might be something like the statement "the natural number n is either even or odd". This certainly holds. Or it might, as in this case, by an identity. Here it is again:

$$x_n = \frac{2^{n+1} - 2}{2^{n+1} - 1}$$

We guessed at this identity in the first part of the question, in the second part we have to actually prove it.

I have mentioned already that the mistake that many students make is in thinking that they have to prove the Principle of Induction for the case in question. This is not what you are supposed to do. On the contrary, you are supposed to take it as given. Indeed, in many systems of formal mathematics it is stated as an axiom and is considered to be self-evident.[2] At this point you should read over it again and convince yourself that it is, if not self evident, then at least plausible. Think about it. If we have proved that $P(k) \Rightarrow P(k+1)$, then if $P(0)$ holds so must $P(1)$, and if $P(1)$ holds so must $P(2)$, and so on, ad infinitum. It is both weird and obvious at the same time. I once heard it described as "free beer tomorrow", but I am not sure that that does it justice. Anyway, the point is that you are supposed to accept it and not to try to prove it.

So if we have the Principle of Induction to hand, how do we use it? At the risk of repetition, we only need to prove that both the base case and the induction step hold and then we are free to assert that $P(n)$ holds, by induction as it were.

The second thing that students tend to get confused about is actually proving the base case and the induction step, the first because it often seems pointless and the second because it is an implication. We tackle the base case $P(0)$ first, or rather $P(1)$, because in this question, as I have already pointed out, we are starting at 1.

In order to prove our identity for the case $n = 1$ then, all we have to do is to substitute 1 into the expression and see that it evaluates to two thirds. By the way, as a quick aside, if you look back

[2] To be more precise, the presence of what is called an inductive set is usually given as an axiom and the Principle of Induction is then derived from it. In passing it is worth noting that the Principle of Induction can in fact be derived directly from the structure of the natural numbers and does not have to depend on the existence of any set.

at the answer I gave, there is some nice notation. The vertical bar with the tiny $n = 1$ at the foot means "evaluated at". I really like it. To continue, the proof is so easy that it is disarming and can leave the student wondering. We know that x_1 is two thirds, they say, so why do we have to prove it? It certainly is the case that x_1 is defined as two thirds but that is not the point. We need to prove that the expression we have guessed previously for x_1 evaluates to two thirds.

Next, how do we prove the induction step, namely the implication $P(k) \Rightarrow P(k+1)$? I do not think the following inference rule is too advanced for this text so I will give it:[3]

$$\frac{[\,A\,] \;\cdots\; B}{A \Rightarrow B}$$

A and B are propositions again. The square brackets around the A mean that it does not hold, we just temporarily suppose that it holds. The three dots signify a derivation. In plain English this rule reads:

"If we suppose that A holds and from that we can derive that B holds, then we say A implies B."

So in order to prove $P(k) \Rightarrow P(k+1)$ we suppose that $P(k)$ holds and from that starting point we derive $P(k+1)$. This is what I did in the answer.

I think it is always a good idea to figure out what $P(n)$ is and state it clearly, because it gives you a handle on things. Also state the Principle of Induction itself in the form I gave in the answer. This will help the examiner to understand that you have some grasp of what is going on. How you derive $P(k+1)$ starting at $P(k)$ is usually the only tricky bit but if you understand and are not daunted by the wider context of the induction proof, you will hopefully stand a good chance.

[3]There is a famous dual to this rule called Modus Ponens, the grand-daddy of inference rules:

$$\frac{A \Rightarrow B \quad A}{B}$$

It reads:

"If A implies B and A holds, then B holds."

We say that Modus Ponens eliminates implication whilst the former rule introduces it. Together they define it.

Question 6

$$\begin{aligned}
\tan(A+B) &= \frac{\sin(A+B)}{\cos(A+B)} \\
&= \frac{\sin A\cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B} \\
&= \frac{\tan A \cos B + \sin B}{\cos B - \tan A \sin B} \\
&= \frac{\tan A + \tan B}{1 - \tan A \tan B}
\end{aligned}$$

$$\therefore \tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

We are given:

$$\begin{aligned}
\tan^2\theta &= 2\tan\theta + 1 \\
1 - \tan^2\theta &= -2\tan\theta
\end{aligned}$$

Substituting into the identity for $\tan 2\theta$ above:

$$\begin{aligned}
\tan 2\theta &= \frac{2\tan\theta}{-2\tan\theta} \\
&= \frac{1}{-1} \\
&= -1
\end{aligned}$$

By substituting θ and 2θ into the identity for $\tan A + B$ above, we get:

$$\tan 3\theta = \frac{\tan\theta + \tan 2\theta}{1 - \tan\theta\tan 2\theta}$$

We are given:

$$\begin{aligned}
\tan\theta &= 2 + \tan 3\theta \\
&= 2 + \frac{\tan\theta + \tan 2\theta}{1 - \tan\theta \tan 2\theta}
\end{aligned}$$

$$\begin{aligned}
\tan\theta(1 - \tan\theta\tan 2\theta) &= 2\tan\theta(1-\tan\theta\tan 2\theta) + \tan\theta + \tan 2\theta \\
\tan\theta - \tan^2\theta\tan 2\theta &= 2 - 2\tan\theta\tan 2\theta + \tan\theta + \tan 2\theta \\
\tan\theta - \tan^2\theta\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) &= 2 - 2\tan\theta\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) + \tan\theta + \left(\frac{2\tan\theta}{1-\tan^2\theta}\right)
\end{aligned}$$

Writing T for $\tan\theta$:

$$T - T^2\left(\frac{2T}{1-T^2}\right) = 2 - 2T\left(\frac{2T}{1-T^2}\right) + T + \left(\frac{2T}{1-T^2}\right)$$

$$T(1-T^2) - T^2(2T) = 2(1-T^2) - 2T(2T) + T(1-T^2) + 2T$$

$$T - T^3 - 2T^3 = 2 - 2T^2 - 4T^2 + T - T^3 + 2T$$

$$0 = 2T^3 - 6T^2 + 2T + 2$$

$$\therefore T^3 - 3T^2 + T + 1 = 0$$

By inspection $T = 1$ is a solution. Dividing by $(T-1)$ leaves $T^2 - 2T - 1 = 0$.

$$T^2 - 2T - 1 = 0$$

$$T^2 - 2T + 1 - 1 - 1 = 0$$

$$(T-1)^2 = 2$$

$$\therefore T = 1 \pm \sqrt{2}$$

Now:

$$\tan\theta = 1 \pm \sqrt{2}$$

$$\tan\theta - 1 = \pm\sqrt{2}$$

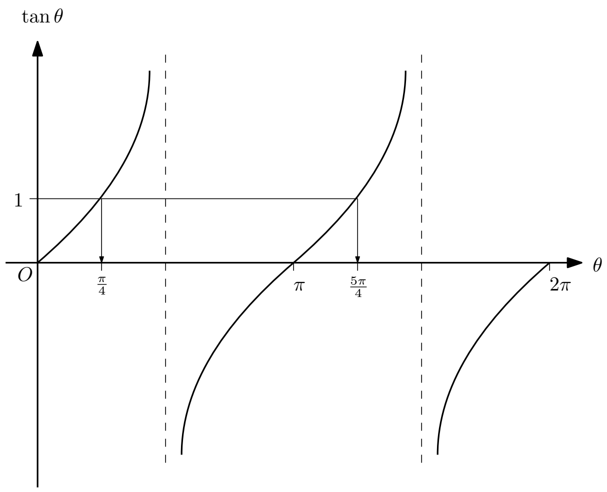
$$\tan^2\theta - 2\tan\theta + 1 = 2$$

$$\tan^2\theta = 2\tan\theta + 1$$

And in this case we proved earlier that $\tan 2\theta = -1$.

The equation $\tan\theta = 1$ on the interval $0 < \theta < 2\pi$ has the following solutions:

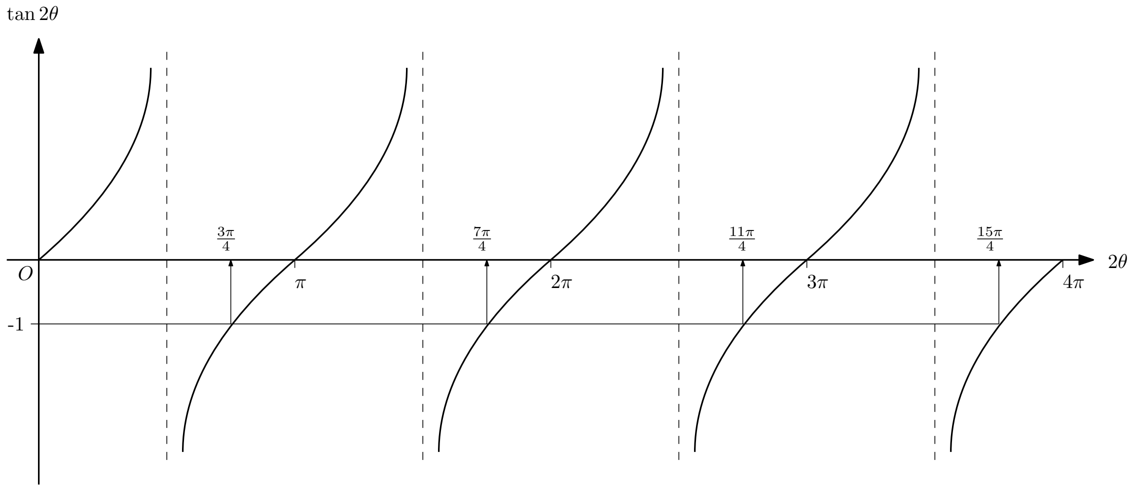
$$\theta = \frac{\pi}{4}, \frac{5\pi}{4}$$



The equation $\tan 2\theta = -1$ on the interval $0 < 2\theta < 4\pi$ has the following solutions:

$$2\theta = \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}, \frac{15\pi}{4}$$

$$\therefore \theta = \frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}, \frac{15\pi}{8}$$



In order to solve $\cot\phi = \cot 3\phi$ we can suppose that the following two equations hold:

$$\cot\phi = \tan\theta$$

$$\cot 3\phi = \tan 3\theta$$

If we consider the graphs of $\tan\theta$ and $\cot\theta$ relative to one another, we can see that the translations amount to the following identity:

$$\cot\psi = \tan(\frac{\pi}{2} - \psi)$$

Therefore the equations above give rise to the following identities:

$$\left.\begin{aligned} \phi &= \tfrac{\pi}{2} - \theta + m\pi \\ 3\phi &= \tfrac{\pi}{2} - 3\theta + n\pi \end{aligned}\right\}$$

Solving, we get a Diophantine equation relating m and n:

$$- \left.\begin{aligned} 3\phi &= 3\tfrac{\pi}{2} - 3\theta + 3m\pi \\ 3\phi &= \tfrac{\pi}{2} - 3\theta + n\pi \end{aligned}\right\}$$

$$0 = \pi + (3m - n)\pi$$

$$3m - n = -1$$

$$m = \frac{1}{3}(n - 1)$$

Since m is an integer, n must be of the form $3k + 1$. Substituting back into the second of the identities:

$$
\begin{aligned}
3\phi &= \frac{\pi}{2} - 3\theta + (3k+1)\pi \\
&= \pi(\frac{1}{2} + 3k + 1) - 3\theta \\
&= \pi(3k + \frac{3}{2}) - 3\theta \\
&= 3\pi(k + \frac{1}{2}) - 3\theta \\
\therefore \phi &= \pi(k + \frac{1}{2}) - \theta
\end{aligned}
$$

Returning to the previous general solutions to the equation $\tan\theta = 2 + \tan 3\theta$ and substituting:

$$
\begin{aligned}
\theta = n\pi + \frac{\pi}{4}: \quad \phi &= (k + \frac{1}{2})\pi - n\pi - \frac{\pi}{4} \\
&= (k-n)\pi - \frac{\pi}{4}
\end{aligned}
$$

Setting $(k-n) = 0, -1$ gives:

$$
\phi = \frac{\pi}{4}, \frac{-3\pi}{4}
$$

$$
\begin{aligned}
\theta = \frac{1}{2}n\pi + \frac{\pi}{8}: \quad \phi &= (k + \frac{1}{2})\pi - \frac{1}{2}n\pi + \frac{\pi}{8} \\
&= \frac{(2k-n)\pi}{2} - \frac{\pi}{8}
\end{aligned}
$$

Setting $(2k-n) = 0, -1, -2, -3$ gives:

$$
\phi = \frac{\pi}{8}, \frac{-3\pi}{8}, \frac{-7\pi}{8}, \frac{-11\pi}{8}
$$

Comments

I know I have stated my aversion to learning things parrot fashion already, but you really should commit these trigonometric identities to memory. Their proof is a straightforward application of Euler's formula and I will give it later, but I still strongly encourage you to learn them:

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$
$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

The reason is that you can derive all the trigonometric identities that you will ever need directly from them. You must become fluent at doing this also. For example, you should be able to derive the identity for $\tan 2\theta$ in no longer than it takes to write the derivation down without pausing. This way, you do not have to remember a large set of identities and can always employ them with confidence.

I found this question heavy going but I have given my workings in full. The first mistake was that although I derived an identity for $\tan 3\theta$, I did not do so solely in terms of $\tan\theta$. This made the subsequent workings more difficult. I got it out, but it took time. The substitution of T for $\tan\theta$ is an essential trick that you should learn, by the way. I probably would have been scuppered without it.

The result of all the workings is a cubic equation. You do not need to learn how to solve cubic equations in general in order to answer any of these questions, by the way. Unless specific instructions are given otherwise, you should always be able to find a root by inspection. You can then use long division to find the quadratic. Try it. I love algebraic long division and can only commend it to you a second time. You will have to forgive me for not typesetting it, though.

Now we come to something that is useful to know. You will note how in order to find the solutions to $\tan 2\theta = -1$, I label the horizontal axis of the graph 2θ and the vertical axis $\tan 2\theta$. Although you have to adjust the range when you do this, it is so much easier to take this approach. Eventually you will have to multiply out or divide through, of course, but you should leave that until the very last step.

On reflection I am not sure whether my next mistake was that much of one but I will walk you through it nonetheless. You can complete the last part of the question more easily if you have the identity for $\tan 3\theta$ in terms of $\tan\theta$ to hand. Using this as a guide, you can derive the corresponding identity for $\cot 3\phi$ in terms of $\cot\phi$ and follow exactly the same reasoning as before. Not having this identity to hand, I did at least realise that I was missing something and tried another approach, namely to figure out how tan and cot are related in terms of transformations. When I figured this out, I was encouraged to see that the different set of inequalities for ϕ were in keeping with these transformations, and so I decided that I was on the right track. On equating $\tan\theta$ with $\cot\phi$, however, again because I did not have the right identity to hand, I made heavy work of showing that $\tan 3\theta$ also equated to $\cot 3\phi$. I guess there is some merit in the approach I took, because you cannot simply assume that the second equality follows from the first.

Moving swiftly one, before covering completing the square I should proffer a couple of suggestions on solving quadratic equations generally. Firstly, unless you can factorise a quadratic immediately by inspection, you should probably complete the square, since it is guaranteed to lead to a result even if the factors turn out to be complex. I recommend that you never bother with the quadratic formula, either. It is pretty cumbersome and is no more than the solution of the general quadratic achieved by, guess what, completing the square. As for the method of completing the square itself, I will lay out the example from the question in some detail:

$$\begin{aligned}
T^2 - 2T - 1 &= 0 \\
T^2 - 2T + 0 - 1 &= 0 \\
T^2 - 2T + (1 - 1) - 1 &= 0 \\
(T^2 - 2T + 1) - 1 - 1 &= 0 \\
(T - 1)^2 - 2 &= 0 \\
\therefore T &= 1 \pm \sqrt{2}
\end{aligned}$$

The trick is the ol' add zero trick, of course, but the gist of it is not that 0 should be replaced with $(1 + 1)$ or that 1 is chosen because it is the square of half the linear coefficient. The gist is the use of the brackets. I always recommend that students write out lines three and four in full, or very nearly, because frequently they fluff the mental calculations if they jump from lines one to five. They usually protest, and then the next time they fluff it, I point out the method again.

This method is particularly useful when the square of half of the linear coefficient happens to be a nasty looking fraction and it also helps to cement the reasoning behind that choice. It is not a bad idea to practise all of this on the aforementioned general form of the quadratic equation. Here are the first few steps to get you started if you want to give it a go:

$$\begin{aligned}
ax^2 + bx + c &= 0 \\
x^2 + \frac{b}{a}x + \frac{c}{a} &= 0 \\
x^2 + \frac{b}{a}x + \left[\left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right] + \frac{c}{a} &= 0 \\
\left[x^2 + 2\left(\frac{b}{2a}\right)x + \left(\frac{b}{2a}\right)^2\right] - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} &= 0 \\
\left[x + \left(\frac{b}{2a}\right)\right]^2 &= \left(\frac{b}{2a}\right)^2 - \frac{c}{a} \\
&\vdots
\end{aligned}$$

Question 7

Substituting t for $\sqrt{1+x^2}+x$:

$$t = (1+x^2)^{\frac{1}{2}} + x$$

$$\frac{dt}{dx} = \frac{1}{2}(1+x^2)^{-}\frac{1}{2}.2x + 1$$

$$= x(1+x^2)^{-\frac{1}{2}} + 1$$

$$\frac{dt}{x(1+x^2)^{-\frac{1}{2}}+1} = dx$$

However, we require:

$$dx = \frac{1}{2}(1+t^{-2})dt$$

Therefore we need to prove:

$$\frac{1}{2}(1+t^{-2}) = \frac{1}{x(1+x^2)^{-\frac{1}{2}}+1}$$

Rearranging:

$$\frac{1}{2}(1+t^{-2}) = \frac{(1+x^2)^{\frac{1}{2}}}{x+(1+x^2)^{\frac{1}{2}}}$$

$$= \frac{(1+x^2)^{\frac{1}{2}}}{t}$$

$$t + t^{-1} = 2(1+x^2)^{\frac{1}{2}}$$

$$t^2 - 2(1+x^2)^{\frac{1}{2}}t + 1 = 0$$

$$t^2 - 2(1+x^2)^{\frac{1}{2}}t + (1+x^2) - (1+x^2) + 1 = 0$$

$$\left[t - (1+x^2)^{\frac{1}{2}}\right]^2 - x^2 = 0$$

$$t - (1+x^2)^{\frac{1}{2}} = \pm x$$

$$t = (1+x^2)^{\frac{1}{2}} \pm x$$

If we take the positive branch, this was our choice for the substitution and, working back up, we arrive at what we needed to prove.

When $x = 0$, $t = 1$ and when $x = \infty$, $y = \infty$, therefore, as required:

$$\int_0^{\infty} f((x^2+1)^{\frac{1}{2}} + x)dx = \frac{1}{2}\int_1^{\infty}(1+t^{-2})f(t)dt$$

In the final part we set $f(t) = t^3$, in which case:

$$
\begin{aligned}
I &= \frac{1}{2}\int_1^\infty (1+t^{-2})t^{-3}dt \\
&= \frac{1}{2}\int_1^\infty t^{-3} + t^{-5}dt \\
&= \frac{1}{2}\left[\frac{t^{-2}}{-2} + \frac{t^{-4}}{-4}\right]_1^\infty \\
&= -\frac{1}{8}\left[2t^{-2} + t^{-4}\right]_1^\infty \\
&= -\frac{1}{8}[2(0)+0] - -\frac{1}{8}[2+1] \\
&= \frac{3}{8}
\end{aligned}
$$

Comments

The substitution must be the one chosen because of what is inside the f. Short of stating it explicitly, which the examiners sometimes actually do, they could hardly have been more helpful.

Finding dx in terms of dt is more difficult. You can see the approach I took. I saw what was required and worked towards it. If the derivation boiled down to an equation governing the substitution, I knew that I would be in the clear. Notice that I did not go to the lengths of reversing the whole derivation in order to arrive at where I started from, if you see what I mean. I think it is enough just to state your method clearly. As ever, two solutions result from solving the quadratic, however in this instance I was free to pick the one I wanted and argue that I could then track back. A nice little question.

Question 8

(i)
$$a^{\ln b} = b^{\ln a}$$
$$\Leftrightarrow \ln\left(a^{\ln b}\right) = \ln\left(b^{\ln a}\right)$$
$$\Leftrightarrow \ln b \ln a = \ln a \ln b$$

True. The second line holds because ln is a strictly increasing function.

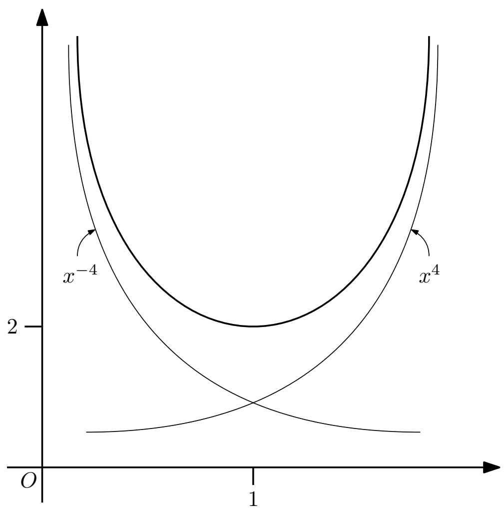
(ii) Setting θ to 0 gives:

$$\cos(\sin 0) = \sin(\cos 0)$$
$$\cos(0) = \sin(1)$$
$$\sin(1) = 1$$

This is false and so the supposition is also false.

(iii) False, because any polynomial $P(x)$ bar a constant one eventually blows up for large enough x. Subtracting $\cos x$ and taking the absolute value does not essentially change that fact. If the polynomial is a constant, because $\cos x$ varies between $+1$ and -1, we cannot restrict the difference between it and any constant to be less than 10^{-6} over all x.

(iv) Consider the graph of $y = x^4 + x^{-4}$:



$$y = x^4 + x^{-4}$$
$$\frac{dy}{dx} = 4x^3 - 4x^{-5}$$
$$\frac{dy}{dx} = 0 : \quad 4x^3 - 4x^{-5} = 0$$
$$x^3 = x^{-5}$$
$$x = \pm 1$$

There is a local minimum at $x = 1$, substituting in gives $y = 2$. From the graph it is clear that the local minimum is global for $x > 0$. Therefore we have $y \geqslant 2$ and thus $y + 3 \geqslant 5$, so true.

Comments

Note that I give an explanation in the first part to the tune of ln being a strictly increasing function. If you are still wondering, draw the graph. I think that there is no need to do any more than state the fact, by the way. You can assume the examiner will give you full credit and would not require any further explanation. My answer to the third part is a bit hand-wavy but I am also pretty sure that this kind of argument will suffice.

Question 9

For angle θ, there are two sides of length $\sin\theta$ and two of length $\cos\theta$. If the perimeter is $P(\theta)$ we have:

$$P(\theta) = 4(\sin\theta + \cos\theta)$$

$$\begin{aligned}
\sin(A+B) &= \sin A\cos B + \cos A\sin B \\
R\sin(\theta' + \theta) &= R\cos\theta'\sin\theta + R\sin\theta'\cos\theta \\
&= \quad 4 \qquad \sin\theta + \quad 4 \qquad \cos\theta
\end{aligned}$$

Comparing coefficients:

$$\left.\begin{aligned}
R\cos\theta' &= 4 \\
R\sin\theta' &= 4
\end{aligned}\right\}$$

$$\begin{aligned}
\frac{R\cos\theta'}{R\sin\theta'} &= \frac{4}{4} \\
\tan\theta' &= 1 \\
\theta' &= \frac{\pi}{4}
\end{aligned}$$

$$\begin{aligned}
R^2\cos^2\theta' + R^2\sin^2\theta' &= 4^2 + 4^2 \\
R^2 &= 32 \\
R &= 4\sqrt{2}
\end{aligned}$$

$$\therefore P(\theta) = 4\sqrt{2}\sin\left(\theta + \frac{\pi}{4}\right)$$

Differentiating and setting to zero to find the maxima and minima:

$$\begin{aligned}
\frac{d}{d\theta}\sin\left(\theta + \frac{\pi}{4}\right) = \cos\left(\theta + \frac{\pi}{4}\right) &= 0 \\
\theta + \frac{\pi}{4} &= \frac{\pi}{2} \\
\theta &= \frac{\pi}{4}
\end{aligned}$$

Differentiating a second time and substituting:

$$\begin{aligned}
\frac{d}{d\theta}\cos\left(\theta + \frac{\pi}{4}\right) &= -\sin\left(\theta + \frac{\pi}{4}\right) \\
-\sin\left(\theta + \frac{\pi}{4}\right)\bigg|_{\theta=\frac{\pi}{4}} &= -\sin\left(\frac{\pi}{2}\right) \\
&= -1
\end{aligned}$$

So a maximum.

For the case $n = 2$ we define the lengths in terms of the coordinates (x, y). In this case the sum of the squares of the lengths is:

$$\begin{aligned}2(2x)^2 + 2(2y)^2 &= 2(4x^2) + 2(4y^2)\\ &= 8(x^2 + y^2)\\ &= 8R^2\end{aligned}$$

So the length of the perimeter remains constant.

For the case $n = 3$ we take $2x$ to x, $2y$ to y and, without loss of generality, also assume that the radius is unity. Making use of Pythagoras we have:

$$\begin{aligned}P &= x^3 + (1 - x^2)^{\frac{3}{2}}\\ P' &= 3x^2 + \frac{3}{2}(1 - x^2)^{\frac{1}{2}}. - 2x\\ &= 3x^2 - 3(1 - x^2)^{\frac{1}{2}}\\ &= 3x\left[x - (1 - x^2)^{\frac{1}{2}}\right]\end{aligned}$$

Setting this to zero gives a stationary point at $x = 0$. Additionally we have:

$$\begin{aligned}x - (1 - x^2)^{\frac{1}{2}} &= 0\\ x &= (1 - x^2)^{\frac{1}{2}}\\ x^2 &= 1 - x^2\\ 2x^2 &= 1\\ x &= \pm\frac{1}{\sqrt{2}}\end{aligned}$$

We can ignore the negative root. Differentiating a second time:

$$\begin{aligned}P'' &= 3\left[2x - \frac{1}{2}(1 - x^2)^{-\frac{1}{2}}\right]\\ P''|_{x=\frac{1}{\sqrt{2}}} &= 3\left[2\frac{1}{2} - \frac{1}{2}\left(1 - \frac{1}{2}\right)^{-\frac{1}{2}}\right]\\ &= 3\left[1 - \frac{1}{2}\left(\frac{1}{2}\right)^{-\frac{1}{2}}\right]\\ &= 3\left[1 - \frac{\sqrt{2}}{2}\right]\\ &= 3\left[1 - \frac{1}{\sqrt{2}}\right]\end{aligned}$$

This is positive, so there is a minimum at $x = \frac{1}{\sqrt{2}}$. On the other hand:

$$\begin{aligned}
P''|_{x=0} &= 3\left[2(0) - \frac{1}{2}(1-0)^{-\frac{1}{2}}\right] \\
&= 3(0-1) \\
&= -3
\end{aligned}$$

So there is a maximum at $x = 0$.

Comments

The first part is another application of the trigonometric identity for $\sin(A + B)$. Notice that I have been very careful in lining up the coefficients. You may have been taught to remember the identities for R and θ parrot fashion when employing this particular variant. I can only advise against this. Again it is worth pointing out that if you master the simple steps and can execute them quickly, you will only have to remember the original identity and will gain in confidence to boot.

Whether to attempt each part of the question with angles or coordinates is down to no more than personal preference, I think. I chose angles for the first part and coordinates for the last. There are couple of little things I did to save myself some time. I changed variables and assumed the radius was unity for the last part. I stated clearly what I was doing, though, so it is fine. I also only wrote P for the perimeter rather than parenthesising it. Again this is fine as long as it is obvious what you are doing.

All of the differentiation and substitution is time consuming and error prone, so I took time to think about each part before plunging in. Luckily for me this worked for the second part. It is very difficult to gauge how much time to give yourself to think before putting pen to paper. I have found in the past that students who do not find normal school maths particularly difficult tend to wade in without giving the question much thought, on the assumption that it will just fall out. Obviously they come unstuck more often that not at this level.

Question 10

$$\begin{aligned}\left(1-\frac{1}{50}\right)^{\frac{1}{2}} &= \left(\frac{49}{50}\right)^{\frac{1}{2}}\\ &= \left(\frac{49\times 2}{100}\right)^{\frac{1}{2}}\\ &= \frac{7\sqrt{2}}{10}\\ \therefore \sqrt{2} &= \frac{10}{7}\left(1-\frac{1}{50}\right)^{\frac{1}{2}}\end{aligned}$$

$$\begin{aligned}(1+x)^n &= 1+\frac{n}{1!}x+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\cdots\\ \frac{10}{7}\left(1-\frac{1}{50}\right)^{\frac{1}{2}} &= \frac{10}{7}\left[1+\frac{1/2}{1!}\left(\frac{-1}{50}\right)+\frac{1/2.-1/2}{2!}\left(\frac{-1}{50}\right)^2+\frac{1/2.-1/2.-3/2}{3!}\left(\frac{-1}{50}\right)^3+\cdots\right]\\ &= \frac{10}{7}\left[1-\frac{1}{100}-\frac{1}{2}\frac{1}{(100)^2}-\frac{1}{2}\frac{1}{(100)^3}+\cdots\right]\\ &= \frac{10}{7}\left[1-\frac{1}{100}-\frac{1}{2}\frac{1}{10,000}-\frac{1}{2}\frac{1}{1,000,000}-\cdots\right]\\ &= 1.428571\left[1-\frac{1}{100}-\frac{1}{2}\frac{1}{10,000}-\frac{1}{2}\frac{1}{1,000,000}-\cdots\right]\\ &\simeq 1.428571-0.014285-\frac{0.000142}{2}-\frac{0.000001}{2}\\ &= 1.414286-0.000071-\cdots\\ &= 1.414215\end{aligned}$$

$$\left(1+\frac{N}{125}\right)^{\frac{1}{3}} = \left(\frac{125+N}{125}\right)^{\frac{1}{3}} = \frac{(125+N)^{\frac{1}{3}}}{5}$$

Choosing $N=3$:

$$\begin{aligned}\left(1+\frac{3}{125}\right)^{\frac{1}{3}} &= \left(\frac{128}{125}\right)^{\frac{1}{3}}\\ &= \frac{(2\times 64)^{\frac{1}{3}}}{5}\\ &= \frac{4}{5}\sqrt[3]{2}\\ \sqrt[3]{2} &= \frac{5}{4}\left(1+\frac{3}{125}\right)^{\frac{1}{3}}\end{aligned}$$

$$\begin{aligned}
\frac{5}{4}\left(1+\frac{3}{125}\right)^{\frac{1}{3}} &= \frac{5}{4}\left[1+\frac{1/3}{1!}\left(\frac{3}{125}\right)+\frac{1/3.-2/3}{2!}\left(\frac{3}{125}\right)^2+\frac{1/3.-2/3.-5/3}{3!}\left(\frac{3}{125}\right)^3+\cdots\right] \\
&= \frac{5}{4}\left(1+\frac{1}{125}-\frac{1}{125^2}+\frac{10}{6}\frac{1}{125^3}+\cdots\right) \\
&= \frac{5}{4}\left(1+\frac{8}{1,000}-\frac{8^2}{1,000^2}+\frac{5}{3}\frac{8^3}{1,000^3}+\cdots\right) \\
&\simeq \frac{5}{4}\left(1+0.008-0.000064\right) \\
&= \frac{5}{4}\times 1.007936 \\
&= \frac{1}{4}\times 5.03968 \\
&= 1.25992
\end{aligned}$$

This only gives 5 decimal places of precision at the very most so we must evaluate the next term:

$$\frac{5\times 5\times 8\times 8\times 8}{4\times 3}=\frac{3200}{3}\simeq 1,067$$

Dividing by a billion moves the decimal point nine places to the left so we have, to six decimal places, 0.000001.

$$\therefore \sqrt[3]{2}=1.259921 \quad \text{(6dp)}$$

Comments

Students have looked at the first calculation and asked me how on earth they were supposed to know, starting from what is given, how to find an expansion for $\sqrt{2}$? I could only explain that, since nothing else came to mind, I just simplified the expression inside the brackets and noted that 49 is a perfect square. From there I noted that 50 is a divisor of two away from another perfect square, namely 100, and then I was away. I think they had a right to feel aggrieved and a hint might have been in order, but it is difficult to see what hint could have been given without giving the game away.

There is also another thing that struck me when I came to type this up, namely, that I arrived at this first solution by not thinking too hard about it and just following my nose, as I have just explained. And I realise that, given the advice I gave in the comments in the previous question about thinking before diving in, this makes me a hypocrite. In fairness, however, I wondered when I made those comments how long it would be before this situation arose. Over-thinking happens, and I should know. The point here is that I could not think of anything at all and so I did the only thing that I could do with what was at hand. The rest was down to luck.

It was within my powers to get the evaluation of the binomial expansion right first time, however the typeset solution makes it look easier than it really is and you are forgiven if you did not get it quite right. You might want to go over it a couple of times. Even though you know the answer, you will gain a little fluency in doing so. That is not bad advice for many of the questions, in fact.

I think the next bit of the question is slightly easier than the corresponding bit of the first part, assuming you even got that far, that is. The last bit is even harder than the corresponding bit of the first part, however, and I did not get it right first time. I multiplied out the $\frac{5}{4}$ earlier and discarded the fourth term in the expansion, not because I thought the first three terms would give me the required accuracy but because, I have to admit, I was worn out by this stage. One star does not do this question justice, it is harder than that. If you got it out in its entirety, you did well.

Question 11

These are the numbers between 0 and 9 inclusive that are not divisible by 2 or 3:

$$1, 3, 7, 9$$

There are 4 numbers and $10 \times 10 = 100$ such groupings, from 0 to 999 inclusive, and since 1000 is divisible by 2 we have:

$$100 \times 4 = 400$$

To find the mean, we group the sum as follows:

$$(1 + 3 + 7 + 9) + (11 + 13 + 17 + 19) + \cdots + (991 + 993 + 997 + 999)$$

We note that $1 + 3 + 7 + 9 = 20$ and again that there are 100 groupings, so we put aside the following:

$$20 \times 100 = 2000$$

This leaves:

$$\begin{aligned}
&(4 \times 0) + (4 \times 10) + \cdots + (4 \times 990) \\
&= 4 \times (0 + 10 + \cdots + 990) \\
&= 4 \times 10 \times (1 + \cdots + 99) \\
&= 40 \times \frac{99 \times 100}{2} \\
&= 198,000
\end{aligned}$$

The sum is therefore $198,000 + 2000 = 200,000$ and the mean:

$$\frac{200,000}{400} = 500$$

For the second part, we note that just as $1000 = 10 \times 10 \times 10$, so $9261 = 21 \times 21 \times 21$, so we arrange the numbers in groups of 21 rather than in groups of 10.

These are the numbers from 0 to 20 inclusive that are not divisible by 3 or 7:

$$1, 2, 4, 5, 8, 10, 11, 13, 16, 17, 19, 20$$

There are 12 numbers and $21 \times 21 = 441$ such groupings so we have:

$$12 \times 441 = 5,292$$

We note that $1 + 2 + 4 + 5 + 8 + 10 + 11 + 13 + 16 + 17 + 19 + 20 = 126$ and that there are 441 groupings this time, so we put aside the following:

$$126 \times 441 = 55,566$$

This leaves:

$$
\begin{aligned}
&(12 \times 0) + (12 \times 21) + \cdots + (12 \times 9,240) \\
&= 12 \times (0 + 21 + \cdots + 9,240) \\
&= 12 \times 21 \times (1 + \cdots + 440) \\
&= 12 \times 21 \times \frac{440 \times 441}{2} \\
&= 6 \times 21 \times 440 \times 441
\end{aligned}
$$

Thus the total is:

$$126 \times 441 + 6 \times 21 \times 440 \times 441 = 441 \times (126 + 6 \times 21 \times 440)$$

And the mean:

$$\frac{441 \times (116 + 6 \times 21 \times 440)}{12 \times 441} = \frac{126 + 6 \times 21 \times 440}{12} = \frac{55,556}{12} = 4,630.5$$

Comments

There is an easy way to calculate sums such as $1+2+\cdots+10$, namely pairing off the first number with the last, the second with the second to last, etc. Then you simply count up the pairs and multiply this number by their constant value. So in this case the answer is $5 \times 11 = 55$. If you are reading this then probably you were one of those who worked all of this out for yourself with very little prompting, and you also probably went on to calculate the sum $1+2+\cdots+100$, say, by the same technique. I will give the general proof later on, by the way.

Having missed the fact that the first part of the question was nothing more than an application of pairing, I repeated the mistake in the second part. However, there is a technique in this all that is worth pointing out. It is an example of not thinking too hard again, in fact. Once I realised that essentially the only difference between the first and second parts of the question was that 10 was replaced by 21, I knew that I could figure it out with hardly any thought. You should have noticed yourself that the structure of the two parts of the question was identical and recognised this as the broadest of hints.

To continue, in order to answer the second part I simply replaced every occurrence of 10 with 21, every 4 with 12, etc and completed the proof in this mechanistic way. The only thing that required any thought at all was realising that the numbers were not going to be so kind this time around and so I had better not multiply out the factors, on the assumption that many of the them would cancel out at the end. I was left with one long division, which I have not shown, otherwise the ploy worked. I think there is a huge amount to be said for behaving like an automaton sometimes. My insistence on being punctilious when completing the square is in a similar vein.

Now on to the general proof for an arithmetic series, which I give because it is surprising how often students cannot give it precisely themselves:[4]

Theorem. The sum of the first n natural numbers is given by the formula:

$$\frac{n(n+1)}{2}$$

Proof. We consider the cases for n even and odd separately.

For n even:

$$\begin{aligned} 1+2+\cdots+(n-1)+n &= [1+n]+[2+(n-1)]+\cdots+\left[\frac{n}{2}+(\frac{n}{2}+1)\right] \\ &= \underbrace{(n+1)+(n+1)+\cdots+(n+1)}_{\frac{1}{2}n \text{ times}} \end{aligned}$$

---

[4]Earlier I gave the inference rules for introducing and eliminating implication. There are actually two rules for introducing disjunction. They are:

$$\frac{A}{A \vee B} \quad \frac{B}{A \vee B}$$

They read:

"If A holds then A or B holds. If B holds then A or B holds."

Not very exciting, but they should make sense. What about the rule for eliminating conjunction? This is more convoluted:

$$\frac{A \Rightarrow C \;\; B \Rightarrow C \;\; A \vee B}{C}$$

I find it interesting that there appears to be no better rule for eliminating disjunction than this one. As far as I know it is also the only rule in propositional logic that requires another previously defined operator, namely implication, in order to work. The reason I bring this up is because it is known as "Proof by cases" as well as its more common name of "Disjunction Elimination". If we set A to "n is even", B is "n is odd" and C to $1+2+\cdots+n = \frac{n(n+1)}{2}$, then the proof here can be seen as an application of this rule.

$$= (n+1) \times \frac{n}{2}$$

$$= \frac{n(n+1)}{2}$$

For n odd:

$$1 + 2 + \cdots + (n-1) + n = [1+n] + [2+(n-1)] + \cdots + \left[\frac{(n-1)}{2} + (\frac{(n-1)}{2} + 1)\right] + \frac{(n+1)}{2}$$

$$= \underbrace{(n+1) + (n+1) + \cdots + (n+1)}_{\frac{1}{2}(n-1) \text{ times}} + \frac{(n+1)}{2}$$

$$= (n+1) \times \frac{n-1}{2} + \frac{(n+1)}{2}$$

$$= \frac{(n+1)(n-1)}{2} + \frac{(n+1)}{2}$$

$$= \frac{(n+1)(n-1+1)}{2}$$

$$= \frac{n(n+1)}{2}$$

Since n is either even or odd, we have for all n:[5]

$$1 + 2 + \cdots + (n-1) + n = \frac{n(n+1)}{2}$$

□

[5] If you want some fun, prove by induction that any natural number n is either even or odd.

Question 12

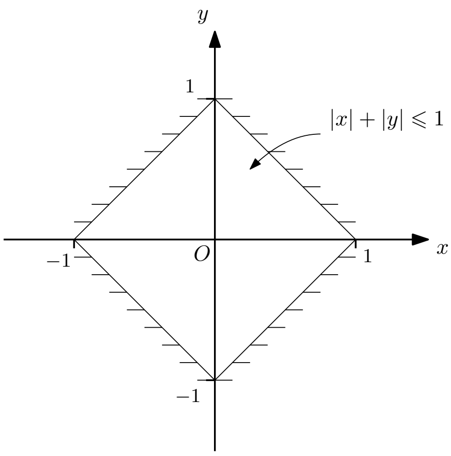
(i)

$$x \geqslant 0, y \geqslant 0: \quad x + y \leqslant 1$$

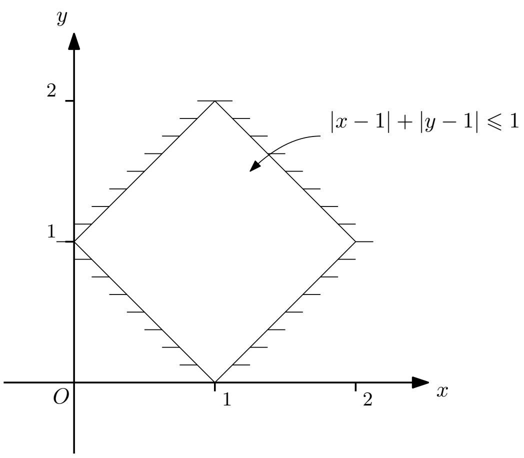
$$x \geqslant 0, y \leqslant 0: \quad x - y \leqslant 1$$

$$x \leqslant 0, y \leqslant 0: \quad -x - y \leqslant 1, \quad x + y \geqslant -1$$

$$x \leqslant 0, y \geqslant 0: \quad -x + y \leqslant 1, \quad x - y \geqslant -1$$



(ii) The region is the same shape but translated by coordinates $(1, 1)$:



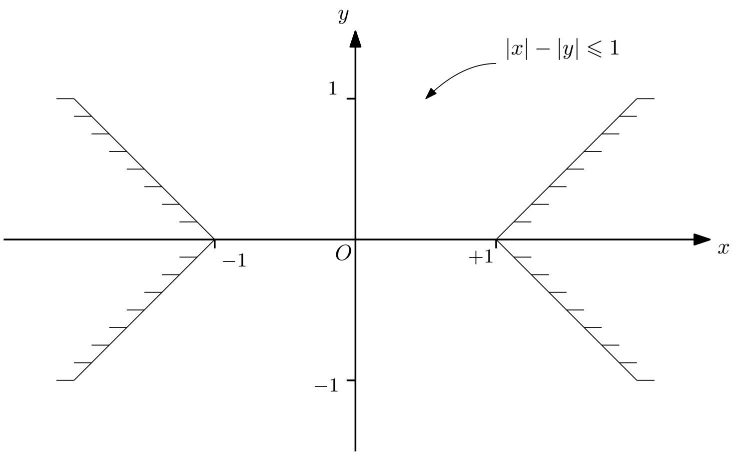
(iii) We consider the region for the simpler inequality $|x| - |y| \leqslant 1$ and then translate afterwards:

$$x \geqslant 0, y \geqslant 0: \quad x - y \leqslant 1, \quad y \geqslant x - 1$$

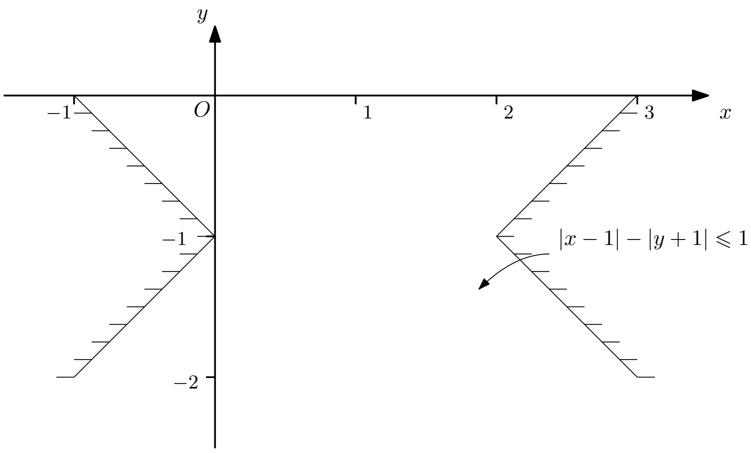
$$x \geqslant 0, y \leqslant 0: \quad x + y \leqslant 1, \quad y \leqslant -x + 1$$

$$x \leqslant 0, y \leqslant 0: \quad -x + y \leqslant 1, \quad y \leqslant x + 1$$

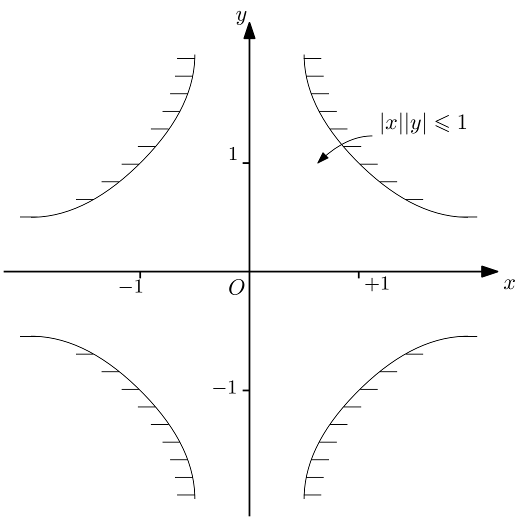
$$x \leqslant 0, y \geqslant 0: \quad -x - y \leqslant 1, \quad y \geqslant -x - 1$$



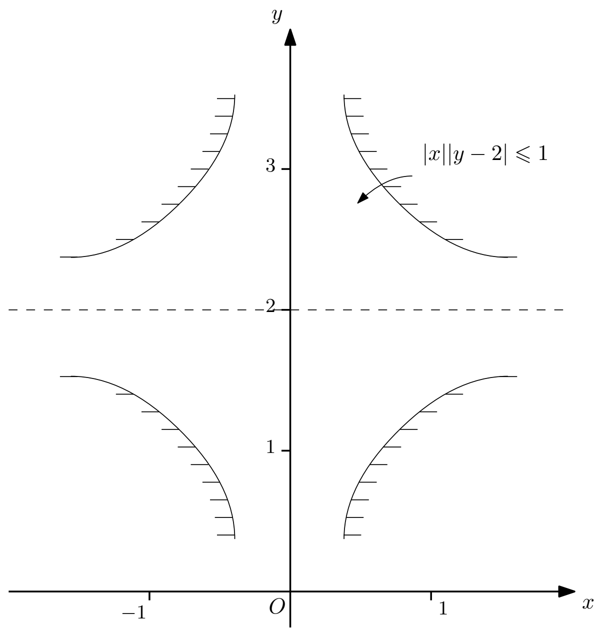
The translation is by coordinates $(1,-1)$ this time:



(iv) We consider a region for the simpler inequality $|x||y| \leqslant 1$ and translate afterwards. For $x \geqslant 0, y \geqslant 0$ the boundary is the unit hyperbola. By symmetry, the region must be the following:



The translation is by coordinates $(0,2)$:



y
$|x||y-2| \leqslant 1$
3
2
1
-1
O
1
x

Comments

There is not much to say about this question. Aside from splitting the inequalities down into branches, which I have already covered, the one technique worth a mention is to treat simpler inequalities and then translate them. If you missed that, perhaps the question was not quite straightforward. The only other thing I can think of is that although the question only asks you to sketch the regions, I gave a little explanation with each. I think this is wise, as is labelling the axes properly.

Question 13

Multiplying out the right hand side:

$$(x-y+2)(x+y+1) = x^2 + xy - x - xy - y^2 + y + 2x + 2y - 2$$
$$= x^2 - y^2 + x + 3y - 2$$

This equals the left hand side and we are done.

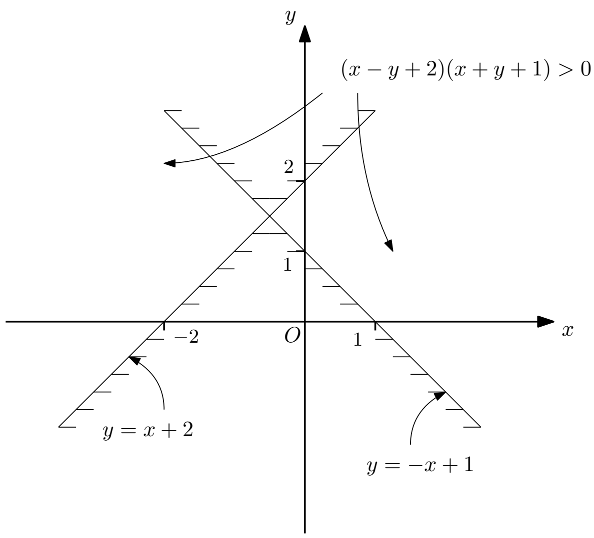
$$x^2 - y^2 + x + 3y > 2$$
$$x^2 - y^2 + x + 3y - 2 > 0$$
$$(x-y+2)(x+y+1) > 0$$

We solve the equality first:

$$(x-y+2)(x+y+1) = 0$$

This gives:

$$x - y + 2 = 0 \quad \text{or} \quad x + y - 1 = 0$$
$$y = x + 2 \quad \text{or} \quad y = -x + 1$$



Substituting $x = 0, y = 0$ into the inequality we get $0 > 2$, which is a contradiction. Substituting $x = 0, y = 3$ into the inequality we again get $0 > 2$. Therefore the points $(0, 0)$ and $0, 3$ lie outside the region. By a similar argument the points $(3, 0)$ and $(-3, 0)$ lie inside the region. Therefore the region is as shown.

For the second part, we try to factorise by inspection. We have:

$$x^2 - 4y^2 + 3x - 2y + 2 = (x + 2y + 2)(x - 2y + 1)$$

Therefore we need to solve the following inequality:

$$(x + 2y + 2)(x - 2y + 1) < 0$$

Again we solve the corresponding equality:

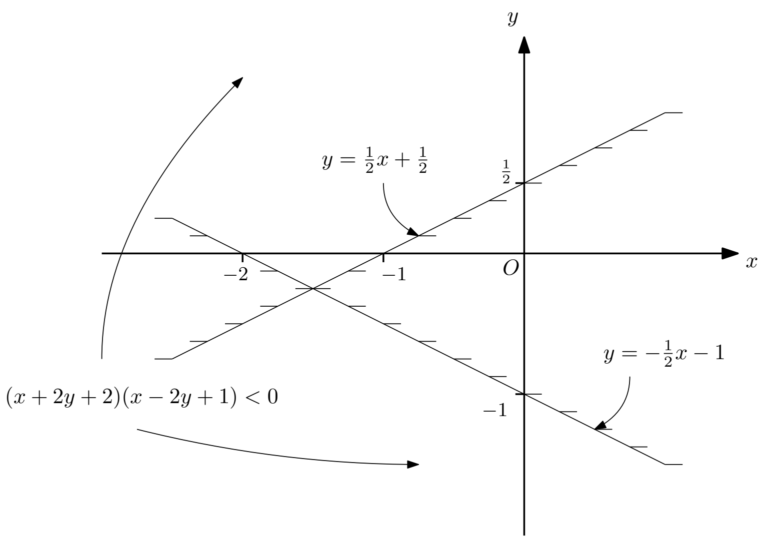
$$(x + 2y + 2)(x - 2y + 1) = 0$$

This gives:

$$x + 2y + 2 = 0 \quad \text{or} \quad x - 2y + 1 = 0$$

$$2y = -x - 2 \quad \text{or} \quad 2y = x + 1$$

$$y = -\frac{1}{2}x - 1 \quad \text{or} \quad y = \frac{1}{2}x + \frac{1}{2}$$



By the same argument as before, we substitute $x = 0, y = 0$ into the equality, which gives $2 < 0$ another contradiction. So the rightmost sub-region is not required and, by symmetry, nor is the left. Therefore region consists of the sub-regions above and below the intersection.

By inspection the point $(0, \frac{3}{2})$ lies inside both regions.

Comments

You should always solve inequalities by solving the corresponding equalities first. There was something of an exception to this rule in the last question, but it still holds. Solving quadratic equalities is the classic case and I feel that it is such an important principle that I will give an example first before tackling the question at hand.

Suppose we wish to solve the following inequality:

$$x^2 + x - 2 < 0$$

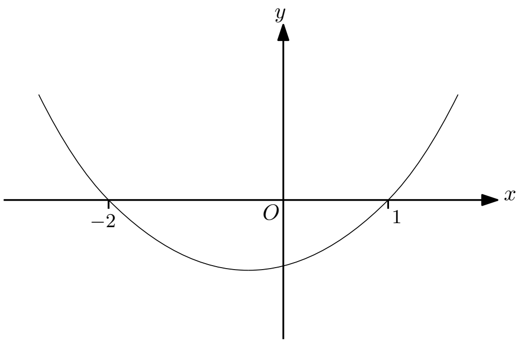
We consider the corresponding equality and factorise:

$$x^2 + x - 2 = 0$$

$$(x-1)(x+2) = 0$$

$$\therefore x = 1 \text{ or } x = -2$$

Then if we sketch the graph...



...we can see directly that the required interval is $-2 < x < 1$.

Without solving the corresponding equality first and sketching the graph, students often resort to factorising in the context of the inequality:

$$(x-1)(x+2) < 0$$

There is nothing wrong with this in itself, however it inevitably leads to the following mistake:

$$(x-1) < 0 \quad \text{or} \quad (x+2) < 0$$

So at the risk of droning on, when faced with solving an inequality, solve the corresponding equality first, and then see where you are.

If you are familiar with the technique of solving inequalities then hopefully getting as far as sketching the graphs with the boundaries of the regions, which are given by the corresponding equalities, was within your grasp. The one thing that remains after this is to figure out where the region lies in each case, and convincing the examiner with a suitable explanation.

The only question really is whether the region comprises the top sub-region together with the bottom sub-region or left together with right. I think it is safe to assume that the examiner does not require more than a nod to the fact that the region cannot lie on both sides of any boundary. The way I chose to argue it was to pick a point in the sub-region and see if it obeyed the inequality. If it did not, I argued that it lay outside of the region and therefore the particular sub-region to which it belonged could not itself form part of the required region. This is a somewhat heuristic argument, but nonetheless a valid one.

By the way, you may be wondering how I managed to factorise $x^2 - 4y^2 + 3x - 2y + 2$ by inspection. To be honest, I wondered this myself when I came to type it up. I realised at the time that there was almost bound to be a connection with the first part of the question and indeed when I looked, I found that the structure of the two inequalities was the same. And since I saw $4y^2$ replacing y^2, I duly swapped y for $2y$ and ended up with something of the following form:

$$(x - 2y + a)(x + 2y + b)$$

The mystery to me when I came to type it up was not how I got this far, but how I figured out a and b in my head. Anyway, here it is in full. Multiplying out:

$$\begin{aligned}(x - 2y + a)(x + 2y + b) &= x^2 + 2xy + bx - 2xy + 4y^2 - 2by + ax + 2ay + ab \\ &= x^2 + 4y^2 + (a + b)x + 2(a - b)y + ab\end{aligned}$$

This gives the following simultaneous equations:

$$\left.\begin{aligned} a + b &= 3 \\ 2(a - b) &= -2 \\ ab &= 2 \end{aligned}\right\}$$

Of course any two of these will do to solve for a and b and it is easy to check that $a = 1$ and $b = 2$.

Question 14

(i) Multiplying through and solving the corresponding equality:

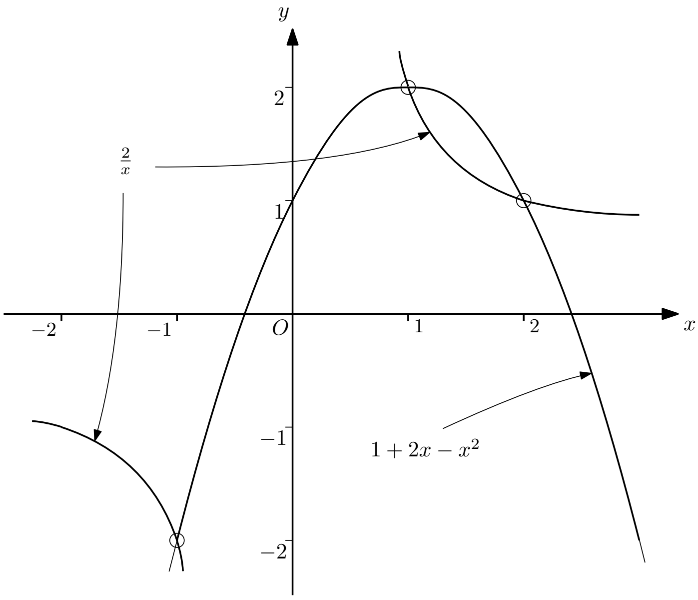
$$x + 2x^2 - x^3 = 2$$

$$x^3 - 2x^2 - x + 2 = 0$$

Putting $x = 2$:

$$8 - 2 \times 4 - 2 + 2 = 0$$

Therefore $x = 2$ is a root. Dividing through gives $x^2 - 1$ and so the other roots are $x = \pm 1$.



We require the value of the quadratic expression to be greater than the reciprocal. From the graph it is clear that this occurs in the following intervals:

$$-1 < x < 0 \quad \text{and} \quad 1 < x < 2$$

(ii)

$$\sqrt{(3x + 10)} > 2 + \sqrt{(x + 4)}$$

$$\sqrt{(3x + 10)} - \sqrt{(x + 4)} > 2$$

$$3x + 10 + x + 4 - 2\sqrt{(3x + 10)(x + 4)} > 4$$

$$4x + 10 - 2\sqrt{\cdots} > 0$$

$$2x + 5 - \sqrt{\cdots} > 0$$

$$2x + 5 > \sqrt{(3x+10)(x+4)}$$

$$(2x+5)^2 > (3x+10)(x+4)$$

$$4x^2 + 20x + 25 > 3x^2 + 22x + 40$$

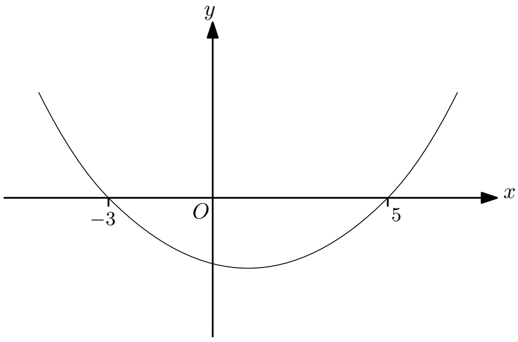
$$x^2 - 2x - 15 > 0$$

Factorising the corresponding equality...

$$x^2 - 2x - 15 = 0$$

$$(x-5)(x-3) = 0$$

...gives roots $x = 3$ and $x = 5$.



Looking at the graph, at first hand the solutions are $x < -3$ and $x > 5$. However, we note that we require $x \geqslant -\frac{10}{3}$, which rules out $x < -3$.

Comments

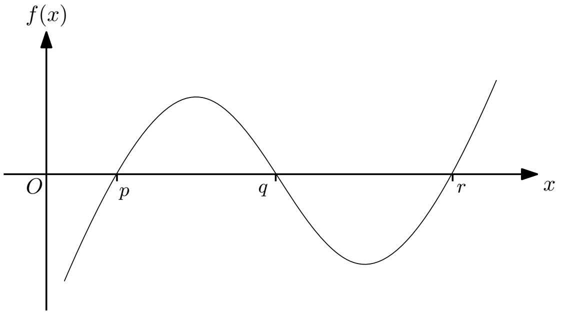
Earlier I commented that most cubic equations will likely be solvable by inspection. This is another example of the examiner's helpfulness.

Note that there are two conditions that have to hold for the second inequality to work for real x, namely $x \geqslant -\frac{10}{3}$ and $x \geqslant -4$. The former encompasses the latter, however, so only the former has to be given. You may be wondering whether you have to be careful with signs when multiplying out square roots in this fashion. Beyond the initial constraints, I think not. By convention the square root sign $\sqrt{}$ stands for the positive square root, so there is no ambiguity in that sense.

I showed the ellipsis $\cdots$ a couple of times because it is what I wrote when I did the question by hand. Doing this would have saved me a few precious seconds in the examination.

Question 15

$$f(x) = (x-p)(x-q)(x-r)$$



$$f(x) = x^3 - (p+q+r)x^2 + (pq+qr+rp)x - pqr$$

$$f'(x) = 3x^2 - 2(p+q+r)x + (pq+qr+rp)$$

Setting $f'(x) = 0$:

$$3x^2 - 2(p+q+r)x + (pq+qr+rp) = 0$$

$$x^2 - \frac{2}{3}(p+q+r)x + \frac{1}{3}(pq+qr+rp) = 0$$

$$x^2 - \frac{2}{3}(p+q+r)x + \frac{1}{9}(p+q+r)^2 - \frac{1}{9}(p+q+r)^2 + \frac{1}{3}(pq+qr+rp) = 0$$

$$\left[x - \frac{1}{3}(p+q+r)\right]^2 = \frac{1}{9}(p+q+r)^2 - \frac{1}{3}(pq+qr+rp)$$

Since the left hand side is a perfect square, we have:

$$\frac{1}{9}(p+q+r)^2 - \frac{1}{3}(pq+qr+rp) \geqslant 0$$

This is the square of the determinant and since $p < q < r$ implies that the roots cannot be equal, we can discard the equality. Therefore we have:

$$\frac{1}{9}(p+q+r)^2 - \frac{1}{3}(pq+qr+rp) > 0$$

$$\frac{1}{9}(p+q+r)^2 > \frac{1}{3}(pq+qr+rp)$$

$$(p+q+r)^2 > 3(pq+qr+rp)$$

Multiplying out...

$$(x^2+gx+h)(x-k) = x^3 - kx^2 + gx^2 - gkx + hx - kh$$

$$= x^3 - (g-k)x^2 + (h-gk)x - kh$$

...and equating coefficients:

$$\left.\begin{aligned} p+q+r &= g-k \\ qr+rp+pq &= h-gk \end{aligned}\right\}$$

Substituting $g-k$ for $p+q+r$ and $h-gk$ for $qr+rp+pq$ into the previous results leads directly to the inequality:

$$(g-k)^2 \geqslant 3(h-gk)$$

We cannot claim that the strict inequality holds because there is no restriction that the turning points are distinct. Also, although we can make use of the substitution in order to save ourselves some working out, we cannot claim that p, q and r are real.

If $g^2 > 4h$ then the quadratic x^2+gx+h has two distinct real roots and therefore the cubic has three distinct real roots. This implies that there are two distinct turning points and the strict inequality holds. However, if $g^2 < 4h$ then the quadratic x^2+gx+h has distinct complex roots and therefore cubic has two distinct complex roots and only real root. This again implies that there are two distinct turning points and again the strict inequality holds.

Comments

Firstly, some advice on drawing curves. Draw them before you put the axes in. This way the axes will not distract you and it is easier to fit the axes to your curve than the other way around. For example, you will find that regardless of the number intersections a cubic curve needs with the x-axis or the position of its intersection with the y-axis, you can always find a set of axes that will fit once you have drawn it. The exception to this rule is that it is best to draw the x-axis when you draw trigonometric curves because, to put it somewhat informally, the parts of the curve above and below the x-axis kind of match up.

Also, do not worry about scale when drawing a curve. If you are required to draw the graphs of $y = x^2$ and $y = 10x^2$, for example, the curves should be the same, only the markings on the axes need to change. If you get over the habit of drawing overly pointy curves just because the coefficients of quadratic and cubic terms happen to be large, for example, you will be much better off. I generally find that students do not take the time to practise drawing curves and I think this is a shame. You can get appreciably better at it with practice and it is also quite gratifying.

If you are not entirely comfortable with the concepts of necessity and sufficiency then do not worry, I suspect they trip up the odd professional mathematician on occasion. Consider the following:

$$A \Rightarrow B$$

This is often paraphrased as "if A holds then B holds", however this is somewhat misleading. Does it mean, for example, that it is necessary for A to hold in order for B to hold? Actually no, it is the other way around, because B could hold anyway. It is necessary for B to hold in order for A to hold because if B does not hold, then we cannot have both A and the implication holding at the same time. Confused? Then just remember that the necessary condition is the consequent in an implication. To be frank, I find sufficiency even more difficult to grasp and tend to avoid thinking about it altogether. I simply treat it as the dual of necessity, and so the sufficient condition is antecedent in an implication. Perhaps this is better:

$$S \Rightarrow N$$

How do we prove that something is not a necessary condition? A plain English example may help. It is necessary for a person to be both male and single in order to be a bachelor, buy it is not necessary for that person to be called Algernon (although with a name like Algernon you would expect bachelorhood to be more likely). Thus we have:

$$\text{bachelor} \Rightarrow \text{male}$$

$$\text{bachelor} \Rightarrow \text{single}$$

$$\neg(\text{bachelor} \Rightarrow \text{Algernon})$$

It should hopefully be a little clearer from these examples that "A implies B" and "B is necessary for A" are effectively synonymous. In order to prove that B is not necessary for A, therefore, we have to show the following:

$$\neg(A \Rightarrow B)$$

Now look at the truth table for implication:[6]

---

[6] You may recall that previously we defined implication by way of the following pair of inference rules:

$$\frac{[\,A\,] \cdots B}{A \Rightarrow B} \qquad \frac{A \Rightarrow B \quad A}{B}$$

Now, however, we have defined it by way of a truth table. These dual approaches to defining logical operators, namely truth tables on the one hand versus pairs of inference rules on the other, pose several subtle questions, not the least of which is the question of equivalence. Both have their uses, but most people would agree that the inference rule approach is more akin to the way mathematicians operate, whether they are aware of it or not, with

A	B	$A \Rightarrow B$
F	F	T
F	T	T
T	F	F
T	T	T

You can see that the only time when $A \Rightarrow B$ is not true is when A is true and B is false. So we need to find a counter-example of the form:

$$A \land \neg B$$

In plain English again, if we want to prove that it is not necessary for a person to be called Algernon in order for them to be a bachelor, we just need to find a bachelor who is not called Algernon.

Returning to the question, we are asked to show that $g^2 > 4h$ is sufficient for $(g-k)^2 > 3(h-gk)$. If we set A to $g^2 > 4h$ and B to $(g-k)^2 > 3(h-gk)$, then we must show $A \Rightarrow B$:

$$\left[g^2 > 4h\right] \Rightarrow \left[(g-k)^2 > 3(h-gk)\right]$$

We are also asked to show that $g^2 > 4h$ is not necessary for $(g-k)^2 > 3(h-gk)$. Being careful how we assign things, if we set A to $(g-k)^2 > 3(h-gk)$ this time and B to $g^2 > 4h$, then we must show that $\neg(A \Rightarrow B)$. In order to do this, we now know that we must find a counter-example of the form $A \land \neg B$:

$$\left[(g-k)^2 > 3(h-gk)\right] \land \neg \left[g^2 > 4h\right]$$

If you read the last paragraph of my answer, you will see that this is what I did. In the second case I showed that you can have $g^2 < 4h$, in fact, but since this implies $\neg(g^2 > 4h)$, I was done.

Personally I think bringing necessity and sufficiency into the question is a bit much, but perhaps it would be difficult to couch things differently without causing even more confusion.

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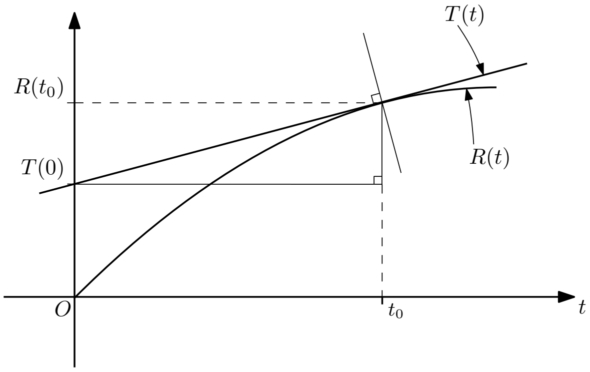
the aforementioned inference rule for introducing implication being the classic example.

If you think about it, a good proportion of the questions here are effectively asking us to prove an implication. We are given some starting point and expected to derive something from it. We are entitled to claim that A implies B only if we can show that B derives from A or, to use fancy language, that B is a syntactic consequence of A. On the other hand, rarely do we wonder about the truth value of any statement. If you look back at my answers and the comments that go along with them, in fact, you will see that I have been careful to avoid the concepts of truth and falsehood. Question 8 was an exception, because it would have been churlish to avoid them there.

The point is that on a day to day basis mathematicians deal with syntax, that is they spend their time deriving things. They seldom care about semantics, that is figuring out what these things evaluate to. It was the genius of Gerhard Gentzen to formalise a system that stressed syntax over semantics, namely Natural Deduction. It enshrines concepts such as suppositions and derivations whilst allowing logical concepts such as implication to remain outside of it and to be defined at a user level, so to speak.

Question 16

(i)



Because $R'(t) > 0$ and $R''(t) < 0$ for $t > 0$, the sketch shows that the tangent must intersect the vertical axis above the origin. That is, if we define $T(t)$ as being the function for the tangent to $R(t)$ at $t = t_0$, then we have:

$$T(0) > 0$$

The gradient T' of $T(t)$ is constant and equal to the gradient of $R(t)$ at $t = t_0$:

$$T' = R'(t_0)$$

The equation of $T(t)$ is given by:

$$\frac{T(t) - T(t_0)}{t - t_0} = T'$$

Substituting $R'(t_0)$ for T' and $R(t_0)$ for $T(t_0)$:

$$\frac{T(t) - R(t_0)}{t - t_0} = R'(t_0)$$

Rearranging:

$$T(t) = (t - t_0)R'(t_0) + R(t_0)$$

Setting $t = 0$:

$$T(0) = (0 - t_0)R'(t_0) + R(t_0) = R(t_0) - t_0R'(t_0)$$

Since $T(0) > 0$:

$$R(t_0) - t_0R'(t_0) > 0$$

Now t_0 is arbitrary and can vary, so we replace it with t:

$$0 < R(t) - tR'(t)$$

$$tR'(t) < R(t)$$

$$t < \frac{R(t)}{R'(t)}$$

$$\therefore t < \frac{1}{H(t)}$$

(ii)

$$
\begin{aligned}
\frac{R'(t)}{R(t)} &= \frac{a}{t} \\
tR'(t) &= aR(t) \\
t\frac{dR}{dt} &= aR \\
\int \frac{1}{R} dR &= a \int \frac{1}{t} dt \\
\ln R &= a \ln t + c \\
e^{\ln R} &= e^{a \ln t + c} \\
R &= e^{a \ln t} e^{c} \\
&= e^{\ln t^a} k \\
R(t) &= kt^a
\end{aligned}
$$

The first condition places no restriction on a. Also:

$$
R'(t) = akt^{a-1} \quad R''(t) = a(a-1)t^{a-2}
$$

Since $k = e^c > 0$ and $t^a > 0$, the second condition requires $a > 0$, and since also $t^{a-2} > 0$, the third condition requires $(a-1) < 0$, that is $a < 1$. So the range is $0 < a < 1$.

(iii)

$$
\begin{aligned}
\frac{dR}{dt} &= \frac{bR}{t^2} \\
\frac{1}{R} dR &= bt^{-2} dt \\
\ln R &= -bt^{-1} + c \\
e^{\ln R} &= e^{-bt^{-1} + c} \\
R(t) &= ke^{-b/t}
\end{aligned}
$$

To begin with $-b/t$ is undefined when $t = 0$ so the first condition cannot be satisfied. Also:

$$
\begin{aligned}
R'(t) &= ke^{-b/t}(-b/t)' \\
&= ke^{-b/t}(-bt^{-1})' \\
&= ke^{-b/t}. -b. -t^{-2} \\
&= \frac{kb}{t^2} e^{-b/t}
\end{aligned}
$$

Since $k = e^c > 0$ still and the other parts of the expression bar b are also strictly positive, we must set $b > 0$ to satisfy the second condition. Differentiating again:

$$
\begin{aligned}
R'(t) &= \left(\frac{kb}{t^2}e^{-b/t}\right)' \\
&= kb\left(t^{-2}e^{bt^{-1}}\right)' \\
&= kb\left(-2t^{-3}e^{bt^{-1}} + t^{-2}e^{bt^{-1}}.bt^{-2}\right) \\
&= kbe^{bt^{-1}}\left(-2t^{-3} + bt^{-4}\right) \\
&= kb\left(-2t + b\right)t^{-4}e^{bt^{-1}}
\end{aligned}
$$

Since we required $b > 0$ to satisfy the second condition, we require $(-2t + b)$ to be negative for all values of t, which is clearly not the case when $2t > b$. Thus the conditions cannot be satisfied.

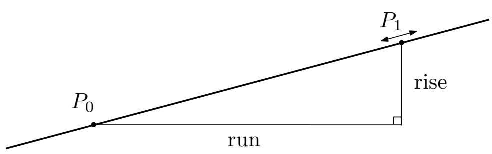
Comments

I often joke with my students that I am the only person who seems to understand what a straight line is. This is borne out of frustration over the fact that whilst they are able to recant equations like $y = mx + c$, they tend to be stumped when these do not work for them. The problem has its root in them not knowing what gradient is. So first of all I get them to write down the following:

$$\text{gradient} = \frac{\text{rise}}{\text{run}}$$

You can argue that this is something of an oversimplification but I do not think so. The standard derivative is no more than the limit of this identity as the run tends to zero, for example.

After dealing with gradients, I proffer that a straight line is no more than a line with a constant gradient. This is obvious, they say. So why can they not apply it? The issue seems to be the lack of any geometric intuition. Here is a straight line with two points on it. If you like you can think of the point P_0 as being stationary and the point P_1 as moving:



If you move P_1 up and down the line, you can see that the triangle changes size. Its bottom edge remains horizontal, however, and its rightmost edge remains vertical. You can imagine this triangle being drawn for any curve. A parabola, say. What distinguishes a straight line from all other curves, indeed uniquely defines it, is the fact that the ratio of the lengths of the sides of the triangle remains constant. This fact does not change even when P_1 passes to the left of P_0, in which case the length of of each side, if we loosely define length and magnitude times direction, becomes negative.

This intuition leads to one of the standard equations given for a straight line. If we give P_1 the coordinates (x_1, y_1), P_0 the coordinates (x_0, y_0) and we denote the gradient by the constant m, then the above identity becomes:

$$\frac{x_1 - x_0}{y_1 - y_0} = m$$

The point is, though, that you really do not even have to remember this. What should guide you is the geometric intuition that the shape of triangle stays the same, it just scales. It may be that one of the points is constant and the other variable, as in the case I have just outlined. Or one of the coordinates of a particular point might be constant whilst the other coordinate is variable. This is what happens when you have to find the intercept of a straight line with an axis, for example. Irrespective of the context, when faced with a problem involving a straight line, always look for the triangle.

In order to solve this question, therefore, I chose a triangle with a constant point at the place where the tangent touches the curve of $R(t)$ and a variable point on the tangent itself. This gave me the following equation:

$$\frac{T(t) - T(t_0)}{t - t_0} = T'$$

After that, I made a couple of substitutions and then extended the triangle until the variable point touched the vertical axis. Finally, by making use of the fact that the intersect with the vertical axis was above the horizontal one, I was home.

The only other thing I shall point out about my workings is that I was, as always, pretty punctilious. I took my time in getting from $\ln R = a \ln t + c$ to $R(t) = kt^a$, for example, because I maintain that combining the four steps, it could actually have been more, is liable only to lead to a mistake that might be hard to track down.

Question 17

Equating the equations for the ellipses gives their intersections:

$$\left.\begin{aligned}(x+2)^2+2y^2&=18\\9(x-1)^2+16y^2&=25\end{aligned}\right\}$$

$$\left.\begin{aligned}8(x+2)^2+16y^2&=144\\9(x-1)^2+16y^2&=25\end{aligned}\right\}$$

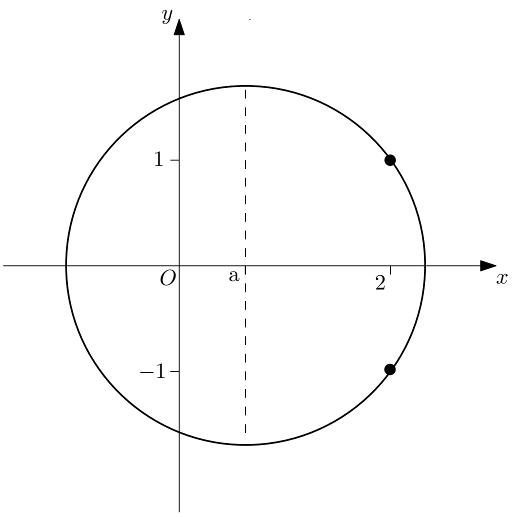
$$\begin{aligned}9(x-1)^2-8(x+2)^2&=-119\\9(x^2-2x+1)-8(x^2+4x+4)&=-119\\9x^2-18x+9-8x^2-32x-32&=-119\\x^2-50x+96&=0\\(x-2)(x-48)&=0\\\therefore x=2,x&=48\end{aligned}$$

Substituting into the first equation:

$$\begin{aligned}x=2:\quad(2+2)^2+2y^2&=18\\4^2+2y^2&=18\\16+2y^2&=18\\2y^2&=2\\y^2&=1\\\therefore y&=\pm1\end{aligned}$$

$$\begin{aligned}x=48:\quad(2+48)^2+2y^2&=18\\50^2+2y^2&=18\\2500+2y^2&=18\\2y^2&=-2482\\y^2&=-1241\\\therefore y\text{ is imaginary}\end{aligned}$$

We need the equation of a circle that passes through the points $(2,1)$ and $(2,-1)$:



The centre of the circle must lie on the x-axis by symmetry. Therefore we must have an equation of the form:

$$(x-a)^2 + y^2 = R^2$$

Every circle must pass through the aforementioned points, so substituting 2 for x and 1 for y^2:

$$(2-a)^2 + 1^2 = R^2$$

Substituting and multiplying out:

$$(x-a)^2 + y^2 = (2-a)^2 + 1^2$$

$$x^2 - 2xa + a^2 + y^2 = 2^2 - 2.2a + a^2 + 1$$

$$\therefore x^2 - 2xa + y^2 = 5 - 4a$$

Comments

There is very little to say about this question. I hope you found it okay. The only thing I can think of is that it helped enormously to realise that there were two distinct parts to it. Firstly, you had to find the equations governing the intersections of the ellipses, and from there the coordinates of the points of intersection themselves. Secondly, with these points to hand, you had to find the general equation of all the circles that pass through them. A little common sense and a peak at the answer helped here. If you conflated these two steps, perhaps the question turned out to be more difficult than the answer suggested.

Notice how the answer is given to you, you only have to find your way to it. Another example of the examiners helpfulness.

Question 18

$$
\begin{aligned}
f(x) &= x^m(x-1)^n \\
f'(x) &= mx^{m-1}(x-1)^n + x^m n(x-1)^{n-1} \\
&= \frac{m}{x}mx^m(x-1)^n + \frac{n}{x-1}x^m n(x-1)^n \\
&= \frac{m}{x}f(x) + \frac{n}{x-1}f(x) \\
&= \left(\frac{m}{x} + \frac{n}{x-1}\right)f(x) \\
&= \left(\frac{m}{x} - \frac{n}{1-x}\right)f(x)
\end{aligned}
$$

Setting $f'(x) = 0$:

$$\left(\frac{m}{x} - \frac{n}{1-x}\right)f(x) = 0$$

$$\therefore \frac{m}{x} - \frac{n}{1-x} = 0 \text{ or } f(x) = 0$$

Suppose $\frac{m}{x} - \frac{n}{1-x} = 0$:

$$
\begin{aligned}
\frac{m}{x} &= \frac{n}{1-x} \\
m(1-x) &= n(x) \\
m &= mx + nx \\
m &= (m+n)x \\
\therefore x &= \frac{m}{m+n}
\end{aligned}
$$

If $m > 1$ then $m > 0$ and if $n > 0$ too then $m + n > 0$, hence:

$$\frac{m}{m+n} > 0$$

Also, if $n > 0$ then $m + n > m$, hence:

$$\frac{m}{m+n} < 1$$

Therefore $0 < x < 1$.

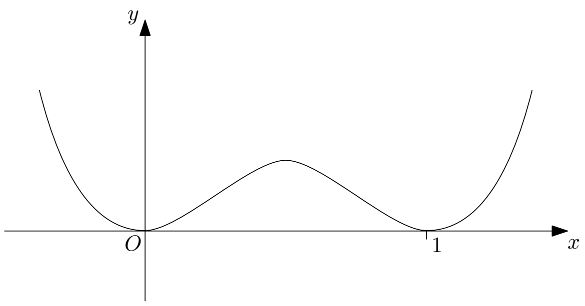
$$f''(x) = \left(\frac{m}{x} - \frac{n}{1-x}\right)' f(x) + \left(\frac{m}{x} - \frac{n}{1-x}\right) f'(x)$$

We know $f'(x) = 0$ at the stationary points, hence:

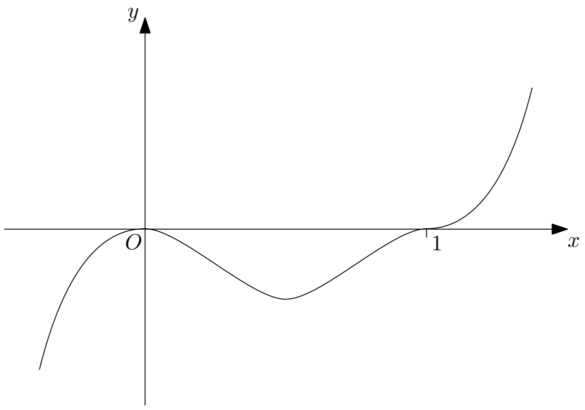
$$
\begin{aligned}
f''(x) &= \left(\frac{m}{x} - \frac{n}{1-x}\right)' f(x) \\
&= \left(\frac{m}{x} - \frac{n}{1-x}\right)' x^m(x-1)^n \\
&= \left(-1.\frac{m}{x^2} - -1.(1-x)'\frac{n}{(1-x)^2}\right) x^m(x-1)^n \\
&= \left(-\frac{m}{x^2} - \frac{n}{(1-x)^2}\right) x^m(x-1)^n \\
&= -\underbrace{\left(\frac{m}{x^2} + \frac{n}{(1-x)^2}\right)}_{>0} x^m(x-1)^n
\end{aligned}
$$

Given that the term inside the brackets is strictly positive, we need to show that $x^m(x-1)^n$ is positive for odd n and negative for even n. This is clearly the case, however, since $x^m > 0$ for $m > 0$ and $0 < x < 1$ and since $(x-1) < 0$ for the same range of x, the sign of $(x-1)^n$ alternates with n, that is $(x-1)^n < 0$ for n odd and $(x-1)^n > 0$ for n even.

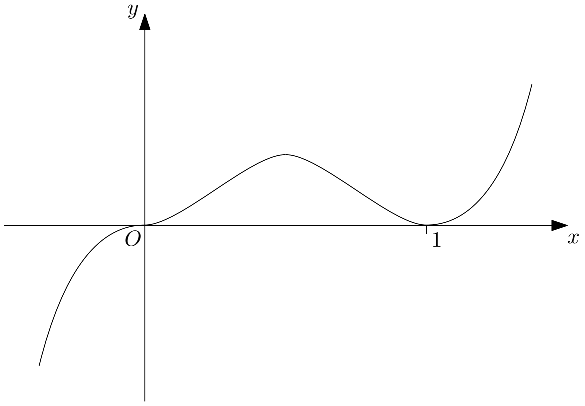
Since the graph of x^2 has the same characteristics as the graph of x^4 and so on, the four cases must correspond to the four combinations of even and odd m and n. Since $F(x) = 0$ and $F'(x) = 0$ for $x = 0$ and $x = 1$, there are turning points at $x = 0$ and $x = 1$ that lie on the x-axis. On the other hand, the stationary point between $x = 0$ and $x = 1$ lies above the x-axis if n is even and below the x-axis if n is odd.



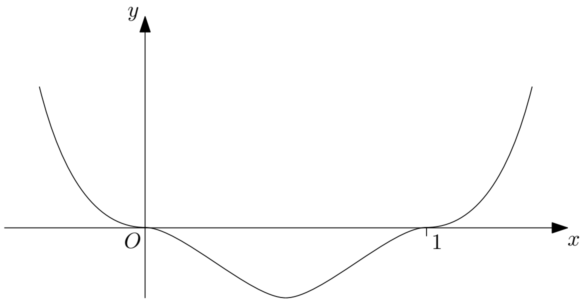
m even, n even



m even, n odd



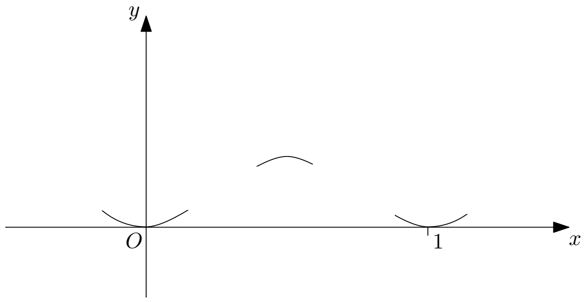
m odd, n even



m odd, n odd

Comments

The technique for drawing graphs such as these is to mark in as many intersections and turning points as possible, also the behaviour as x tends to positive or negative infinity, and then join everything up. Below shows the idea for the first graph, where both m and n are even. Rather than work out the type of the turning points at $x = 0$ and $x = 1$, I simply noted that the x^m term dominates around $x = 0$ and the $(x-1)^n$ term dominates around $x = 1$.



If either the answer I gave or these comments give the impression that drawing these kinds of graphs is necessarily always easy, be assured that getting it completely right all the time is not.

I thought after typing this up that I should give some justification for the observation that the x^m term dominates around the origin and so on. One way to see this is to use the binomial expansion:

$$\begin{aligned} x^m(x-1)^n &= (-1)^n x^m (1-x)^n \\ &= (-1)^n x^m \left[1 + \frac{n}{1!}(-x) + \frac{n(n-1)}{2!}(-x)^2 + \cdots\right] \\ &= (-1)^n x^m \left[1 + O(x)\right] \end{aligned}$$

Since we are interested in the behaviour as $|x| \to 0$, the $O(x)$ term becomes negligible and the function is roughly $(-1)^n x^m$, as expected. To see that the $(x-1)^n$ term dominates around $x = 1$, you would first substitute z for $x - 1$ and then use exactly the same argument for z around 0.

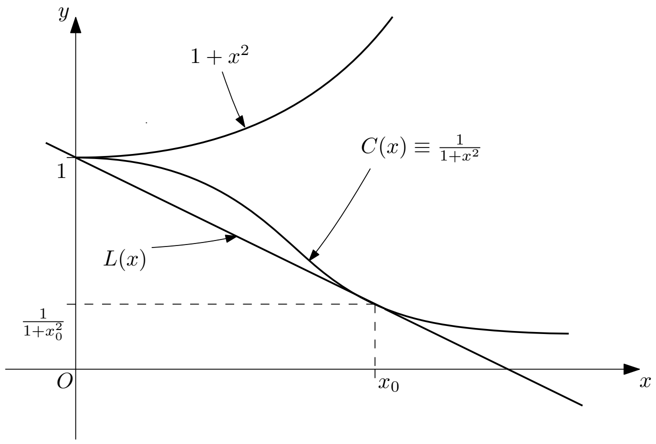
By the way, the $O(x)$ term, which reads "order x", is "big O" notation. You might want to google it if you have not come across it before.

Question 19

$$y = (1+x^2)^{-1}$$

$$y' = -2x(1+x^2)^{-2}$$

$$\therefore\ y'|_{x=0} = 0$$



Using x_0 rather than x for the x-coordinate of the point when the straight line is tangent to the curve, the gradient of the straight line $L(x)$ is the constant $C'(x_0)$, therefore the equation is:

$$C'(x_0) = \frac{L(x)-1}{x-0}$$

$$-\frac{2x_0}{(1+x_0^2)^2} = \frac{L(x)-1}{x}$$

$$-\frac{2xx_0}{(1+x_0^2)^2} = L(x)-1$$

$$L(x) = 1 - \frac{2xx_0}{(1+x_0^2)^2}$$

To find x_0 we note that as well as $C'(x_0) = L'(x_0)$ we also have $C(x_0) = L(x_0)$:

$$\frac{1}{1+x_0^2} = 1 - \frac{2x_0^2}{(1+x_0^2)^2}$$

$$1+x_0^2 = (1+x_0^2)^2 - 2x_0^2$$

$$= 1 + 2x_0^2 + (x_0^2)^2 - 2x_0^2$$

$$x_0^2 = (x_0^2)^2$$

$$0 = (x_0^2)^2 - x_0^2$$

$$0 = x_0^2(x_0^2 - 1)$$

Since $x_0 > 0$, the only solution is $x_0 = 1$. This gives:

$$\begin{aligned} L(x) &= 1 - \frac{2x.1}{(1+1^2)^2} \\ &= 1 - \frac{2x}{4} \\ &= 1 - \frac{1}{2}x \end{aligned}$$

If we drop the restriction that $x_0 > 0$ we can see that if we had substituted x for x_0 in the derivation above, then the only other solution when we equated $L(x)$ and $C(x)$ would have been $x = 0$. Therefore there are no further intersections. The area under L for $0 \leqslant x \leqslant 1$ is the sum of the areas of the enclosed rectangle and triangle:

$$\int_0^1 L(x)dx = (1 \times \frac{1}{2}) + \frac{1}{2}(1 \times \frac{1}{2}) = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

We are given that the area under the curve is $\pi/4$, therefore using the obvious inequality that comes from comparing the areas:

$$\begin{aligned} \frac{\pi}{4} &> \frac{3}{4} \\ \therefore \pi &> 3 \end{aligned}$$

The volume of revolution is given by the following:

$$V = \int_a^b 2\pi xy dx$$

For the curve we have:

$$\begin{aligned} V_C &= \int_0^1 2\pi \frac{x}{1+x^2} dx \\ &= \pi \int_0^1 \frac{2x}{1+x^2} dx \\ &= \pi \left[\ln(1+x^2) \right]_0^1 \\ &= \pi(\ln 2 - ln 1) \\ &= \pi \ln 2 \end{aligned}$$

For the straight line we have:

$$\begin{aligned} V_L &= \int_0^1 2\pi x \left(1 - \frac{1}{2}x\right) dx \\ &= 2\pi \int_0^1 x - \frac{x^2}{2} dx \\ &= 2\pi \left[\frac{x^2}{2} - \frac{x^3}{6} \right]_0^1 \end{aligned}$$

$$= 2\pi\left(\left[\frac{1}{2} - \frac{1}{6}\right] - [0 - 0]\right)$$

$$= \frac{2\pi}{3}$$

Since $V_C > V_L$, we have:

$$\pi \ln 2 > \frac{2\pi}{3}$$

$$\therefore \ln 2 > \frac{2}{3}$$

Comments

This question left me chasing my tail a bit early on. Perhaps it is just sour grapes on my part but I think it could have been worded a little better. What threw me was the question's stipulation that x-coordinate of the point where the straight line and curve meet be given as, well, x. In the end I plucked up the courage to call it x_0, but thought it wise to point out that I had done so. It is also one of those questions that require two curves, one of them a straight line in this instance, to be drawn on the same set of axes, hence y also gets overloaded. I used the functions $L(x)$ and $C(x)$ rather than the overloaded y and strongly recommend that you do the same in these situations. It is surprising how difficult and time consuming a question becomes when your variables get muddled.

Once I had sorted out my variable names, aside from the lingering feeling that I had transgressed in some way by doing so, the answer begun to take shape. However, I managed to prove that there is only one intersection between the curve and the straight line for $x > 0$ in the process of still finding an equation for the latter. I re-used the proof when the time came, or, rather, I suggested that x be substituted for x_0 in the proof and left it at that.

There are several things that you need to know about differentiation and integration for this question, so I will go through them one by one. This first thing I will point out is the integral that the examiner gives you. This is kind of them, but you should know how to prove it.

$$I = \int_0^1 \frac{1}{1+x^2} dx$$

We substitute $\tan\theta$ for x:

$$x = \tan\theta$$

$$\frac{dx}{d\theta} = \sec^2\theta$$

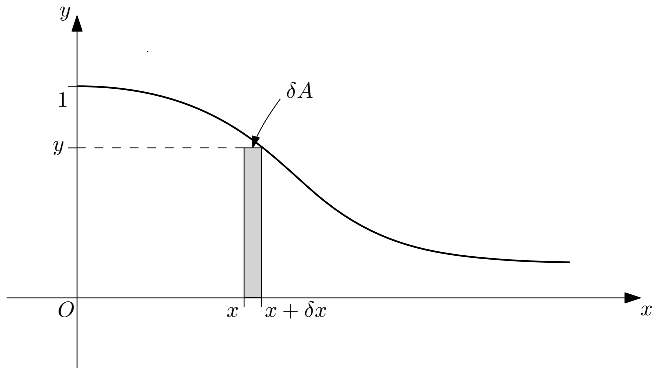
$$dx = \sec^2\theta d\theta$$

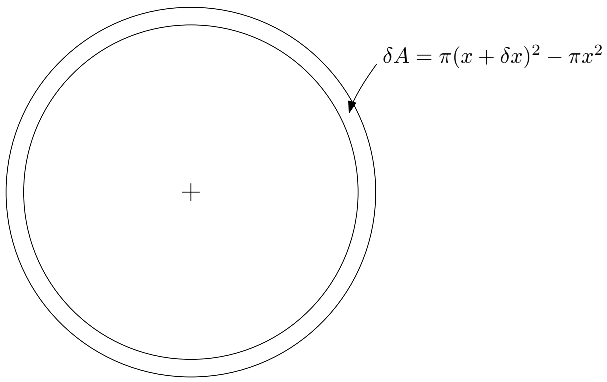
When $x = 0$, $\theta = 0$ and when $x = 1$, $\theta = {}^{\pi}/_{4}$.

$$\begin{aligned} I &= \int_0^{\pi/4} \frac{1}{1+\tan^2\theta} \sec^2\theta d\theta \\ &= \int_0^{\pi/4} \frac{1}{\sec^2\theta} \sec^2\theta d\theta \\ &= \int_0^{\pi/4} d\theta \\ &= [\theta]_0^{\pi/4} \\ &= \frac{\pi}{4} \end{aligned}$$

I used two further identities in this proof, namely $\sec^2\theta = 1 + \tan^2\theta$ and $\tan'\theta = \sec^2\theta$. For the first you just make use of the identity $\sin^2\theta + \cos^2\theta = 1$ and divide through by $\cos^2\theta$. The second one requires the quotient rule, which I will come back to.

The other identity that is made use of is volume of revolution. It surprised me a little that it was not given, but here is a rough proof anyway. I really like it, I suppose because I worked it out at the time off the cuff, which is what you were asked to do.





Consider the rectangle of width δx and height y, as shown in the graph. If we take the top side and rotate it around the y-axis, it inscribes an annulus with inner radius x and outer radius $x+\delta x$. Its area is the difference of the areas of the outer and inner circles:

$$\begin{aligned}\delta A &= \pi(x + \delta x)^2 - \pi x^2 \\ &= \pi(x^2 + 2x\delta x + \delta x^2) - \pi x^2 \\ &= \pi(x^2 + 2x\delta x + \delta x^2 - x^2) \\ &= 2\pi x\delta x\end{aligned}$$

If we multiply this by the height y of the rectangle, we get the volume of the annular cylinder:

$$\delta V = 2\pi xy\delta x$$

The volume of revolution we take as the limit of the sum of these volumes:

$$\begin{aligned}V &= \lim_{\delta x \to 0} \sum \delta V \\ &= \lim_{\delta x \to 0} \sum 2\pi xy\delta x \\ &= 2\pi \lim_{\delta x \to 0} \sum xy\delta x \\ &= 2\pi \int_0^1 xydx\end{aligned}$$

This is not a rigorous proof by any means but it is more than enough at this level.

There are two other identities that are made use of in the answer and their proofs are two of my favourites. The first is the aforementioned quotient rule. There is some sleight of hand with the use of the product rule as you will see:

$$
\begin{aligned}
h &= \frac{f}{g} \\
hg &= f \\
(hg)' &= f' \\
h'g + hg' &= f' \\
h'g &= f' - hg' \\
h' &= \frac{f' - hg'}{g} \\
h' &= \frac{f' - (f/g)g'}{g} \qquad \therefore h' = \frac{f'g - fg'}{g^2}
\end{aligned}
$$

The second is the integral of the reciprocal. Again there is some sleight of hand, this time in the way that the derivative of the exponential function is employed:

$$
\begin{aligned}
y &= \ln x \\
e^y &= x \\
\frac{d}{dy}(e^y) &= \frac{dx}{dy} \\
e^y &= \frac{dx}{dy} \\
x &= \frac{dx}{dy} \\
\frac{1}{x} &= \frac{dy}{dx} \\
\int \frac{1}{x} dx &= \int dy \\
&= y + c \\
\therefore \int \frac{1}{x} dx &= \ln x + c
\end{aligned}
$$

I never tire of these proofs. I hope you enjoyed them. The last proof can in fact be generalised to an arbitrary function, which is the form that is needed to find the integral of $\tan\theta$:

$$\int \frac{f'(x)}{f(x)} dx = \ln f(x) + c$$

To prove this you just substitute $f(x)$ for x. Reusing x in this way is perhaps not very helpful but you should be able to work it out. Just substitute z for x in the integral of the reciprocal and then substitute $f(x)$ for z.

Question 20

$$I + J = \int_0^a \frac{\cos x + \sin x}{\sin x + \cos x} dx \qquad I - J = \int_0^a \frac{\cos x - \sin x}{\sin x + \cos x} dx$$

$$= \int_0^a dx \qquad = [\ln(\sin x + \cos x)]_0^a$$

$$= [x]_0^a \qquad = \ln(\sin a + \cos a) - \ln(\sin 0 + \cos 0)$$

$$= a - 0 \qquad = \ln(\sin a + \cos a) - \ln 1$$

$$= a \qquad = \ln(\sin a + \cos a)$$

$$\therefore 2I = (I + J) + (I - J) = a + \ln(\sin a + \cos a)$$

(i) We define I and J in a similar vein:

$$I = \int_0^{\pi/2} \frac{\cos x}{p \sin x + q \cos x} dx \quad J = \int_0^{\pi/2} \frac{\sin x}{p \sin x + q \cos x} dx$$

$$qI + pJ = \int_0^{\pi/2} \frac{q \cos x + p \sin x}{p \sin x + q \cos x} dx \qquad pI - qJ = \int_0^{\pi/2} \frac{p \cos x - q \sin x}{p \sin x + q \cos x} dx$$

$$= \int_0^{\pi/2} dx \qquad = [\ln(p \sin x + q \cos x)]_0^{\pi/2}$$

$$= \frac{\pi}{2} \qquad = \ln\left(p \sin \frac{\pi}{2} + q \cos \frac{\pi}{2}\right) - \ln\left(\sin 0 + q \cos 0\right)$$

$$= \ln p - \ln q$$

$$q(qI + pJ) + p(pI - qJ) = q\frac{\pi}{2} + p \ln p - p \ln q$$

$$q^2 I + p^2 I = q\frac{\pi}{2} + p \ln \frac{p}{q}$$

$$\therefore I = \frac{1}{p^2 + q^2}\left(q\frac{\pi}{2} + p \ln \frac{p}{q}\right)$$

(ii) Again we define I and J similarly:

$$I = \int_0^{\pi/2} \frac{\cos x + 4}{3 \sin x + 4 \cos x + 25} dx \quad J = \int_0^{\pi/2} \frac{\sin x + 3}{3 \sin x + 4 \cos x + 25} dx$$

$$
\begin{aligned}
4I + 3J &= \int_0^{\pi/2} \frac{4\cos x + 3\sin x + 25}{3\sin x + 4\cos x + 25} dx \\
&= \int_0^{\pi/2} dx \\
&= \frac{\pi}{2}
\end{aligned}
$$

$$
\begin{aligned}
3I - 4J &= \int_0^{\pi/2} \frac{3\cos x - 4\sin x}{3\sin x + 4\cos x + 25} dx \\
&= [\ln(3\sin x + 4\cos x)]_0^{\pi/2} \\
&= \ln\left(3\sin\frac{\pi}{2} + 4\cos\frac{\pi}{2} + 25\right) - \ln\left(3\sin 0 + 4\cos 0 + 25\right) \\
&= \ln\frac{28}{29}
\end{aligned}
$$

$$
\begin{aligned}
4(4I + 3J) + 3(3I - 4J) &= 4\frac{\pi}{2} + 3\ln\frac{28}{29} \\
25I &= 2\pi + 3\ln\frac{28}{29}
\end{aligned}
$$

$$
\therefore I = \frac{1}{25}\left(2\pi + 3\ln\frac{28}{29}\right)
$$

Comments

You might have noticed that once again the examiner has been kind in that not only are you given the overall method, you also got the answer to the first part.

If you get the right identities for I and J in each case, everything should fall out. If you do not, it can be maddening. When I attempted this question with a student, we tried the following identities for the second part:

$$I = \int_0^{\pi/2} \frac{q \cos x}{p \sin x + q \cos x} dx \quad J = \int_0^{\pi/2} \frac{p \sin x}{p \sin x + q \cos x} dx$$

That is, we placed q and p inside the integrals rather than multiplying I and J by them respectively. As a result, rather than $I - J$ you need the following:

$$\frac{p}{q} I - \frac{q}{p} J = \int_0^{\pi/2} \frac{p \cos x - q \sin x}{p \sin x + q \cos x} dx$$

This does work and we got the answer out in the end, but why I did not adjust the coefficients to recover the obvious symmetry is beyond me. Furthermore, this method does not work for the third part, where you have to both adjust the numerators by adding constants and adjust I and J by multiplying by others. I think we gave up in the end.

Question 21

(i)

$$\begin{aligned}
f(x) &= x^4 - 4ax^3 + 6b^2x^2 - 4c^3x + d^4 \\
&= (x-\alpha)(x-\beta)(x-\gamma)(x-\delta) \\
&= x^4 - (\alpha+\beta+\gamma+\delta)x^3 \\
&\qquad + (\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)x^2 \\
&\qquad - (\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)x \\
&\qquad + \alpha\beta\gamma\delta
\end{aligned}$$

From the given identity:

$$\begin{aligned}
\frac{1}{4}(\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta) &> (\alpha\beta\gamma.\alpha\beta\delta.\alpha\gamma\delta.\beta\gamma\delta)^{1/4} \\
\cdots &> (\alpha^3\beta^3\gamma^3\delta^3)^{1/4} \\
\cdots &> (\alpha\beta\gamma\delta)^{3/4} \\
\frac{1}{4}(4c^3) &> (d^4)^{3/4} \\
c^3 &> d^3
\end{aligned}$$

Since c and d are both positive, we have $c > d$.

(ii)

$$\begin{aligned}
f'(x) = 4x^3 &- 3(\alpha+\beta+\gamma+\delta)x^2 \\
&+ 2(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)x \\
&- (\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)
\end{aligned}$$

We write $f'(x)$ as follows:

$$\begin{aligned}
f'(x) &= 4(x-p)(x-q)(x-r) \\
&= 4\left[x^3 - (p+q+r)x^2 + (pq+pr+qr)x - pqr\right] \\
&= 4x^3 - 4(p+q+r)x^2 + 4(pq+pr+qr)x - 4pqr
\end{aligned}$$

Since the roots of the cubic are distinct and real, there are three distinct turning points and therefore the roots of the derivative are also distinct and real. To continue, using the given identity again:

$$\begin{aligned}
\frac{1}{3}(pq+pr+qr) &> (pq.pr.qr)^{1/3} \\
\cdots &> (p^2q^2r^2)^{1/3} \\
\cdots &> (pqr)^{2/3}
\end{aligned}$$

Equating coefficients:

$$\left.\begin{aligned} 4(pq+pr+qr) &= 2(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta) \\ 4pqr &= (\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta) \end{aligned}\right\}$$

$$\left.\begin{aligned} (pq+pr+qr) &= \tfrac{1}{2}(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta) \\ (pqr) &= \tfrac{1}{4}(\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta) \end{aligned}\right\}$$

Substituting into the inequality:

$$\frac{1}{3}.\frac{1}{2}(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta) > \left(\frac{1}{4}(\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)\right)^{2/3}$$

$$\frac{1}{6}(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta) > \left(\frac{1}{4}(\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)\right)^{2/3}$$

$$\frac{1}{6}(6b^2) > \left(\frac{1}{4}(4c)^3\right)^{2/3}$$

$$b^2 > c^2$$

Since b and c are both positive, we have $b > c$.

(iii)

$$\begin{aligned} f(x) &= x^4 - 4ax^3 + 6b^2x^2 - 4c^3x + d^4 \\ f'(x) &= 4x^3 - 12ax^2 + 12b^2x - 12c^3 \\ f''(x) &= 12x^2 - 24ax + 12b^2 \end{aligned}$$

By the same argument as before, the second derivative has two distinct, real roots, hence:

$$\begin{aligned} f''(x) &= 12(x-s)(x-t) \\ &= 12\left[x^2 - (s+t)x + st\right] \\ &= 12x^2 - 12(s+t) + 12st \end{aligned}$$

Equating coefficients again we get $12(s+t) - 24a$ and $12(st) = 12b^2$. Then applying the inequality:

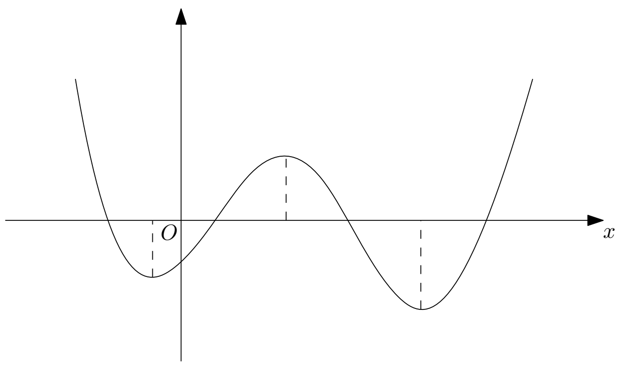
$$\frac{1}{2}(2a) > (b^2)^{1/2}$$

Since $a > 0$ and $b > 0$ we must have $a > b$.

Comments

A good question but quite a long one, especially if, like me, you did not always take the quickest route. As is often the case, the examiners have been kind and instruct you carefully in what to do. They stop short of telling you to make use of the identity but since it is given, you obviously have to.

The second part is the tricky part. You have to realise that you cannot blithely apply the identity a second time, because in doing so you will always get an inequality with d on the left. Once again, however, the examiner helps you by instructing you to take the derivative. Note the language. This is not a "hence or otherwise" question, you are given specific instructions and therefore you had better take them. You also need to realise, and it would be wise to state, that if a quartic equation has four real, distinct roots, then the derivative has three distinct, real ones. A simple sketch of the graph of such a quartic is all you need to see this:



I made a meal of the second part, not because I did not get the gist of it, fortunately I did, but because I used the form of the equation with the four roots α, β, γ and δ. Consequently I had a lot of careful substituting to do. My punctiliousness saved me but I had to work hard. I was pretty exhausted after this but I at least realised my mistake and so the third part, although it uses the exact same arguments, is significantly shorter.

Question 22

$$
\begin{aligned}
F_0 &= 2^{2^0} + 1 \\
&= 2^1 + 1 \\
&= 3 \\
F_1 &= 2^{2^1} + 1 \\
&= 2^2 + 1 \\
&= 5
\end{aligned}
$$

$$
\begin{aligned}
F_2 &= 2^{2^2} + 1 \\
&= 2^4 + 1 \\
&= 17 \\
F_3 &= 2^{2^3} + 1 \\
&= 2^7 + 1 \\
&= 257
\end{aligned}
$$

$k = 1$:

$$
\begin{aligned}
F_0 &= F_1 - 2 \\
3 &= 5 - 2 \ \checkmark
\end{aligned}
$$

$k = 2$:

$$
\begin{aligned}
F_0F_1 &= F_2 - 2 \\
3 \times 5 &= 17 - 2 \ \checkmark
\end{aligned}
$$

$k = 3$:

$$
\begin{aligned}
F_0F_1F_2 &= F_3 - 2 \\
3 \times 5 \times 17 &= 257 - 2 \ \checkmark
\end{aligned}
$$

We define $P(k)$ to be $F_0F_1 \cdots F_{k-1} = F_k - 2$. We have already proved the base case $P(1)$, so it only remains to prove the induction step $P(j) \Rightarrow P(j+1)$.

$$
\begin{aligned}
F_0F_1 \cdots F_j &= (F_0F_1 \cdots F_{j-1})F_j \\
&= (F_j - 2)F_j \\
&= F_j^2 - 2F_j \\
&= (2^{2^j} + 1)^2 - 2(2^{2^j} + 1) \\
&= (2^{2^j})^2 + 2(2^{2^j}) + 1 - 2(2^{2^j}) - 2 \\
&= (2^{2^j})^2 + 1 - 2 \\
&= 2^{2 \times 2^j} + 1 - 2 \\
&= 2^{2^{j+1}} + 1 - 2 \\
&= F_{j+1} - 2
\end{aligned}
$$

Suppose for some $K > 1$ that $K|F_i$ and $K|F_j$. Without loss of generality we set $i > j$. Then:

$$\begin{aligned}\frac{F_j - 2}{K} &= \frac{F_0 F_1 \cdots F_j F_{j+1} \cdots F_{i-1}}{K} \\ &= F_0 F_1 \cdots \frac{F_j}{K} F_{j+1} \cdots F_{i-1}\end{aligned}$$

Therefore $K|(F_j - 2)$. If $K|F_j$ and $K|(F_j - 2)$ then $K|(F_j - (F_j - 2))$ or $K|2$. Therefore $K = 2$, contradicting both K_i and K_j being odd.

We decompose each Fermat number into primes. Since no two Fermat numbers have a prime factor in common, their compositions can have no prime factor in common. Therefore the product of the first n Fermat numbers contains at least n primes. Since there are infinitely many Fermat numbers, therefore, there must be infinitely many primes.

Comments

You may have wondered when you started this question what on earth was going on, it seems far too easy to start with. It gets more challenging as you go along, however.

One thing I noticed with the induction proof is that the proposition is indexed with k rather than n. Given that students who are not confident with induction proofs always seem to mix up their k's and n's, this is perhaps not very helpful. I thought of switching the k to an n before continuing but then thought it best just to use another variable for the induction step, namely j. I did not take my own earlier advice, namely to actually write the two steps that an induction proof requires in their most abstract form first. Remember previously I advised that you should always write down that in order to prove $P(n)$ you need:

1. $P(1)$
2. $P(k) \Rightarrow P(k+1)$ for all $k \geqslant 1$

Here I should have written down that in order to prove $P(k)$ you need:

1. $P(1)$
2. $P(j) \Rightarrow P(j+1)$ for all $j \geqslant 1$

Hopefully you can see what I mean by mention of all these n's, k's and j's now. I also did not mention anywhere during the derivation where I actually made use of the supposition $P(j)$, but I think this is fine, just.

The next proof that no two Fermat numbers share a common factor greater than 1 I would say is hard. I just wrote down the identity that I was previously asked to prove and followed my nose. It is definitely one of those occasions when students ask, how was I supposed to get that?

The last student who questioned me on that part of the question got his own back and managed the last proof with aplomb before I had even gathered my thoughts. I was gratified in being able to work it out just now as I came to type it up, however. It is a nice result.

Question 23

$$\frac{2n-1}{n(n+1)(n+2)} = \frac{A}{n} + \frac{B}{n+1} + \frac{C}{n+2}$$

$$= \frac{A(n+1)(n+2) + Bn(n+2) + Cn(n+1)}{n(n+1)(n+2}$$

$$2n-1 = A(n^2+3n+2) + B(n^2+2n) + C(n^2+n)$$

$$= n^2(A+B+C) + n(3A+2B+C) + (2a)$$

$$\left.\begin{aligned} A+B+C &= 0 \\ 3A+2B+C &= 2 \\ 2A &= -1 \end{aligned}\right\}$$

So $A = {}^{-1}/_{2}$. This leaves:

$$\left.\begin{aligned} B+C &= {}^{1}/_{2} \\ 2B+C &= {}^{7}/_{2} \end{aligned}\right\}$$

So $B = 3$ and $C = -{}^{5}/_{2}$. Therefore:

$$\frac{2n-1}{n(n+1)(n+2)} = \frac{1}{2}\left(-\frac{1}{n} + \frac{6}{n+1} - \frac{5}{n+2}\right)$$

Writing out the terms...

$$\begin{aligned} S_N = &\frac{1}{2}\left(-\frac{1}{1} + \frac{6}{2} - \frac{5}{3}\right) \\ &+ \frac{1}{2}\left(-\frac{1}{2} + \frac{6}{3} - \frac{5}{4}\right) \\ &+ \frac{1}{2}\left(-\frac{1}{3} + \frac{6}{4} - \frac{5}{5}\right) \\ &\vdots \\ &+ \frac{1}{2}\left(-\frac{1}{N-1} + \frac{6}{N} - \frac{5}{N+1}\right) \\ &+ \frac{1}{2}\left(-\frac{1}{N} + \frac{6}{N+1} - \frac{5}{N+2}\right) \end{aligned}$$

...we see that the diagonals cancel. We are left with:

$$\begin{aligned} S_N &= \frac{1}{2}\left(-\frac{1}{1} + \frac{6}{2} - \frac{1}{2} - \frac{5}{N+1} + \frac{6}{N+1} - \frac{5}{N+2}\right) \\ &= \frac{1}{2}\left(\frac{3}{2} + \frac{1}{N+1} - \frac{5}{N+2}\right) \end{aligned}$$

$$
\begin{aligned}
\lim_{N\to\infty} S_N &= \lim_{N\to\infty} \frac{1}{2}\left(\frac{3}{2} + \frac{1}{N+1} - \frac{5}{N+2}\right) \\
&= \frac{1}{2}.\frac{3}{2} + \lim_{N\to\infty} \frac{1}{2}\frac{1}{N+1} - \lim_{N\to\infty} \frac{1}{2}\frac{5}{N+2} \\
&= \frac{3}{4}
\end{aligned}
$$

$$
\begin{aligned}
\frac{a_n}{a_1} &= \frac{a_n}{a_{n-1}}.\frac{a_{n-1}}{a_{n-2}}.\cdots\frac{a_2}{a_1} \\
&= \frac{(n-1)(2n-1)}{(n+2)(2n-3)} \times \frac{(n-2)(2n-3)}{(n+1)(2n-5)} \times \frac{(n-3)(2n-5)}{n(2n-7)} \times \cdots \frac{(2)(5)}{(5)(3)} \times \frac{(1)(3)}{(4)(1)} \\
&= \frac{n-1}{n+2}.\frac{n-2}{n+1}.\frac{n-3}{n}.\frac{n-4}{n-1}.\cdots\frac{4}{7}.\frac{3}{6}.\frac{2}{5}.\frac{1}{4} \times \frac{2n-1}{2n-3}.\frac{2n-3}{2n-5}.\frac{2n-5}{2n-7}.\cdots\frac{5}{3}.\frac{3}{1} \\
&= \frac{3.2.1}{(n+2)(n+1)n}.\frac{2n-1}{1} \\
&= \frac{6(2n-1)}{n(n+1)(n+2)}
\end{aligned}
$$

Hence if we write S_N as $s_0 + s_1 + \cdots s_N$ then we have:

$$
\frac{a_n}{a_1} = 6s_n
$$

Finally:

$$
\begin{aligned}
\sum_{n=1}^{\infty} a_n &= a_1 \sum_{n=1}^{\infty} \frac{a_n}{a_1} \\
&= 6a_1 \sum_{n=1}^{\infty} s_n \\
&= \lim_{N\to\infty} 6a_1 S_N \\
&= 6a_1 \lim_{N\to\infty} S_N \\
&= 6.\frac{2}{9}.\frac{3}{4} \\
&= 1
\end{aligned}
$$

Comments

The diagonals I mention go from top left to bottom right. I am afraid I chose not to spend the time to figure out how to type them up. Similarly for the expanded product later on. For this product, you will note that I spilt out the fractions into two separate products and went quite a way before the cancelling become obvious. I learned something while doing this myself, actually, which is that it is best to spend a little extra time writing out a few more fractions rather than convincing yourself of what cancels with what in your head.

Apropos of the diagonalisation, this was another one of those how was I supposed to know questions. This was surprising at the time because the student had realised that the correct way to start off was to find the partial fractions and indeed had done so. I seem to remember that they could not prove the formula for a geometric series and I put this down as the root cause. Anyway, here it is:

$$\begin{aligned} A_n &= a + ar + ar^2 + \cdots ar^{n-1} \\ rA_n &= ar + ar^2 + \cdots ar^{n-1} + ar^n \\ rA_n - A_n &= ar^n - a \\ A_n(r-1) &= a(r^n - 1) \\ A_n &= \frac{r^n - 1}{r - 1} \quad r \neq 1 \end{aligned}$$

It is a diagonalisation argument, of course. In my opinion it is not quite up there with the proofs of the quotient rule or the integral of the reciprocal, but it is still a nice proof. Again I have omitted to draw the diagonal lines, apologies for this.

Question 24

(i)

$$(m-3)^3 + m^3 = (m+3)^3$$

$$m^3 - 9m^2 + 27m - 27 + m^3 = m^3 + 9m^2 + 27m + 27$$

$$m^3 - 18m^2 - 54 = 0$$

$$m^2(m-18) = 54$$

$$m - 18 = \frac{54}{m^2}$$

If m is an integer, m^2 must be a factor of 54.

We look at all of the factors of 54:

$$1, 2, 3, 6, 9, 18, 27, 54$$

Since m^2 is a square number, it must be 1 or 9. Either of these choices leads to a negative left hand side and since the right hand side is positive, neither works. Therefore m cannot be an integer.
(ii)

$$(n-6)^3 + n^3 = (n+6)^3$$

$$n^3 - 18n^2 + 108n - 216 + n^3 = n^3 + 18n^2 + 108n + 216$$

$$n^3 - 36n^2 = 432$$

$$n^2(n-36) = 432$$

$$n^3 - 36n^2 = 12 \times 36$$

$$n^3 = 36(12 + n^2)$$

$$n^3 = 2 \times 18(12 + n^2)$$

Since n^3 is even, n is even. Writing $n = 2m$:

$$(2m-6)^3 + 2m^3 = (2m+6)^3$$

$$2^3(m-3)^3 + 2^3m^3 = 2^3(m+3)^3$$

$$(m-3)^3 + m^3 = (m+3)^3$$

We already know that there is no such m, and so there is no such n.

(iii)

$$(n-a)^3 + n^3 = (n+a)^3$$

$$n^3 - 3an^2 + 3a^2n - a^3 + n^3 = n^3 + 3an^2 + 3a^2n + a^3$$

$$n^3 - 6an^2 = 2a^3$$

$$n^3 = 2(a^3 + 3an^2)$$

Hence n^3 is even and therefore n must be even. Setting $n = 2k$:

$$(2k)^3 - 6a(2k)^2 = 2a^3$$

$$8k^3 - 24ak^2 = 2a^3$$

$$4(2k^3 - 6ak^2) = 2a^3$$

$$2(2k^3 - 6ak^2) = a^3$$

Hence a^3 is even and therefore a must be even. Setting $a = 2b$:

$$(2k - 2b)^3 + (2k)^3 = (2k + 2b)^3$$

$$(k - b)^3 + (k)^3 = (k + b)$$

Therefore we arrive at where we started in the fact that both k and b must be even and so, since the process can only be repeated a finite number of times, we will eventually arrive at a point where either a descendent of n or a descendent of a must be odd but still satisfy the above equation, which is a contradiction.

To argue more convincingly, we know that both n and a have to be even. Therefore they can be written $n \equiv n_i = 2^i r$ and $a \equiv a_j = 2^j r$, where $i, j \geqslant 1$ and r, s are both odd. In other words, we can only divide both n and a by 2 a finite number of times before the result is an odd number, even if that odd number happens to be one. From above, we know that if n_i and a_j satisfy the equation then n_{i-1} and a_{j-i} must also satisfy the equation. However, we know that neither n_0 and k_0, being both odd, satisfy the equation, and so whichever we arrive at first results in a contradiction.

Comments

This was hopefully a straightforward question for you aside possibly from the last part. The technique of the proof is perhaps not too hard to discern but it is difficult to find the words. I had two stabs at it. I then took a sneaky look at the model answer to convince myself that I had done enough and in fact the first of my efforts would have sufficed. I would not like to have to formalise it.

Question 25

The area under the curve defined by $f(t)$ between 0 and 1 is less than the area of the rectangle between 0 and 1 of height K.

$$\begin{aligned}\frac{1}{n(n-t)} &= \frac{A}{n} + \frac{B}{n-t} \\ &= \frac{A(n-t)+Bn}{n(n-t)} \\ &= \frac{(A+B)n - At}{n(n-t)}\end{aligned}$$

$$\therefore A = -1, B = 1$$

$$\begin{aligned}\int_0^1 \frac{1}{n(n-t)} dt &= \int_0^1 \frac{1}{n-t} dt - \int_0^1 \frac{1}{n} dt \\ &= I_n - \frac{1}{n}\int_0^1 dt \\ &= I_n - \frac{1}{n}\end{aligned}$$

Substituting s for $n-t$ in I_n:

$$ds = -dt$$

$$\begin{aligned}I_n &= \int_n^{n-1} \frac{1}{s}. - ds \\ &= \int_{n-1}^{n} \frac{1}{s} ds \\ &= [\ln s]_{n-1}^{n} \\ &= \ln\left(\frac{n}{n-1}\right)\end{aligned}$$

$$\therefore \int_0^1 \frac{1}{n(n-t)} = \ln\left(\frac{n}{n-1}\right) - \frac{1}{n}$$

Setting the integrand to $f(t)$, that is...

$$f(t) = \frac{1}{n-t} - \frac{1}{n}$$

...and making use of the initial argument gives the required inequality.

Taking the finite sum:

$$\sum_{n=2}^{N} 0 \leqslant \sum_{n=2}^{N} \ln\left(\frac{n}{n-1}\right) - \frac{1}{n} \leqslant \sum_{n=2}^{N} \frac{1}{n-1} - \frac{1}{n}$$

$$0 \leqslant \sum_{n=2}^{N} \ln\left(\frac{n}{n-1}\right) - \sum_{n=2}^{N} \frac{1}{n} \leqslant \sum_{n=2}^{N} \frac{1}{n-1} - \frac{1}{n}$$

$$0 \leqslant \sum_{n=2}^{N} [\ln n - \ln(n-1)] - \sum_{n=2}^{N} \frac{1}{n} \leqslant \sum_{n=2}^{N} \left[\frac{1}{n-1} - \frac{1}{n}\right]$$

$$0 \leqslant [\ln 2 - \ln 1] + [\ln 3 - \ln 2] + \cdots + [\ln N - \ln(N-1)] - \sum_{n=2}^{N} \frac{1}{n} \leqslant \left[\frac{1}{1} - \frac{1}{2}\right] + \left[\frac{1}{2} - \frac{1}{3}\right] + \cdots \left[\frac{1}{N-1} - \frac{1}{N}\right]$$

$$0 \leqslant -\ln 1 + \ln N - \sum_{n=2}^{N} \frac{1}{n} \leqslant 1 - \frac{1}{N}$$

$$0 \leqslant \ln N - \sum_{n=2}^{N} \frac{1}{n} \leqslant 1$$

Taking the right hand inequality:

$$\ln N - \sum_{n=2}^{N} \frac{1}{n} \leqslant 1$$

$$\ln N - 1 \leqslant \sum_{n=2}^{N} \frac{1}{n}$$

Since $\ln N \to \infty$ as $N \to \infty$ on the left, so must the sum on the right.

Taking the left hand inequality:

$$0 \leqslant \ln N - \sum_{n=2}^{N} \frac{1}{n}$$

$$\sum_{n=2}^{N} \frac{1}{n} \leqslant \ln N$$

$$1 + \sum_{n=2}^{N} \frac{1}{n} \leqslant \ln N + 1$$

$$\sum_{n=1}^{N} \frac{1}{n} \leqslant \ln N + 1$$

$$
\begin{aligned}
N &= 10^{30} \\
\ln N &= \ln 10^{30} \\
&= \ln(10^3)^{10} \\
&= 10 \ln 10^3 = 10 \ln 1000 < 10 \ln 1024 = 10 \ln 2^{10} = 100 \ln 2 < 100
\end{aligned}
$$

Thus $\ln N + 1 < 101$ and we are done.

Comments

I liked this question, very mathsy. The last part might have proved tricky. You have to guess that the right approach is to write 10^{30} as $(10^3)^{10}$ out of a few possibilities. I got it right second time around. The first thing I tried was to make use of the fact that 1024 divided by 10 is just the right side of 101, which got me nowhere.

Question 26

$$y = \frac{ax+b}{cx+d}$$

$$\frac{dy}{dx} = \frac{(cx+d)a-(ax+b)c}{(cx+d)^2}$$

$$= \frac{acx+ad-acx-bc}{(cx+d)^2}$$

$$= \frac{ad-bc}{(cx+d)^2}$$

$$I = \int_0^1 \frac{1}{(x+3)^2}\ln\left(\frac{x+1}{x+3}\right)dx$$

We set:

$$y = \frac{x+1}{x+3}$$

Therefore:

$$\frac{dy}{dx} = \frac{1\times 3-1\times 1}{(x+3)^2}$$

$$= \frac{2}{(x+3)^2}$$

$$(x+3)^2 dy = 2dx$$

$$dx = \frac{(x+3)^2}{2}dy$$

$$I = \int_{1/3}^{1/2} \frac{1}{(x+3)^2}\ln y\frac{(x+3)^2}{2}dy$$

$$= \frac{1}{2}\int_{1/3}^{1/2} \ln y\ dy$$

$$= \frac{1}{2}\left[y(\ln y-1)\right]_{1/3}^{1/2}$$

$$= \frac{1}{2}\left[\frac{1}{2}(\ln\frac{1}{2}-1)\right] - \frac{1}{2}\left[\frac{1}{3}(\ln\frac{1}{3}-1)\right]$$

$$= \frac{1}{4}\ln\frac{1}{2}-\frac{1}{4}-\frac{1}{6}\ln\frac{1}{3}+\frac{1}{6}$$

$$= \frac{1}{6}\ln 3-\frac{1}{4}\ln 2-\frac{1}{12}$$

We made use of the identity:

$$\int_a^b \ln y\ dy = [y(\ln y-1)]_a^b$$

This comes from the following:

$$
\begin{aligned}
(y \ln y)' &= y' \ln y + y(\ln y)' \\
&= 1. \ln y + y. \frac{1}{y} \\
&= \ln y + 1 \\
[y \ln y] &= \int \ln y \; dy + [y] \\
\int \ln y \; dy &= [y \ln y] - [y] \\
&= [y(\ln y - 1)]
\end{aligned}
$$

To continue:

$$
\begin{aligned}
I &= \int_0^1 \frac{1}{(x+3)^2} \ln\left(\frac{x^2+3x+2}{x+3}\right) dx \\
&= \int_0^1 \frac{1}{(x+3)^2} \ln\left(\frac{(x+1)(x+2)}{x+3}\right) dx \\
&= \int_0^1 \frac{1}{(x+3)^2} \ln\left(\frac{x+1}{x+3}\right) dx + \int_0^1 \frac{1}{(x+3)^2} \ln\left(\frac{x+2}{x+3}\right) dx \\
&= I_1 + I_2
\end{aligned}
$$

I_1 is our previous I. I_2 is identical bar that b is 2 this time, not 1:

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{1 \times 3 - 1 \times 2}{(x+3)^2} \\
&= \frac{1}{(x+3)^2} \\
dx &= (x+3)^2 dy
\end{aligned}
$$

The limits also change. Instead of $1/3$ and $2/4$ we have $2/3$ and $3/4$. Therefore:

$$
\begin{aligned}
I_2 &= [y(\ln y - 1)]_{2/3}^{3/4} \\
&= \frac{3}{4} \ln \frac{3}{4} - \frac{3}{4} - \frac{2}{3} \ln \frac{2}{3} + \frac{2}{3}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{3}{4} \ln 3 - \frac{3}{4} \ln 4 - \frac{3}{4} - \frac{2}{3} \ln 2 + \frac{2}{3} \ln 3 + \frac{2}{3} \\
&= -\frac{3}{4} \ln 4 + \frac{17}{12} \ln 3 - \frac{2}{3} \ln 2 - \frac{1}{12}
\end{aligned}
$$

Finally:

$$
\begin{aligned}
I &= I_1 + I_2 \\
&= \left[\frac{1}{6}\ln 3 - \frac{1}{4}\ln 2 - \frac{1}{12}\right] + \left[-\frac{3}{4}\ln 4 + \frac{17}{12}\ln 3 - \frac{2}{3}\ln 2 - \frac{1}{12}\right] \\
&= -\frac{3}{4}\ln 4 + \frac{19}{12}\ln 3 - \frac{11}{12}\ln 2 - \frac{1}{6} \\
&= \frac{1}{12}\left(-9\ln 4 + 19\ln 3 - 11\ln 2 - 2\right) \\
&= \frac{1}{12}\left(19\ln 3 - 29\ln 2 - 2\right)
\end{aligned}
$$

Lastly, we note the following:

$$
\begin{aligned}
\ln\left(\frac{x+1}{x+2}\right) &= \ln\left(\frac{x+1}{x+3}.\frac{x+3}{x+2}\right) \\
&= \ln\left(\frac{x+1}{x+3}\right) - \ln\left(\frac{x+2}{x+3}\right)
\end{aligned}
$$

This gives the integrals I_1 and I_2, only the difference this time, not the sum:

$$
\begin{aligned}
I_1 - I_2 &= \left[\frac{1}{6}\ln 3 - \frac{1}{4}\ln 2 - \frac{1}{12}\right] - \left[-\frac{3}{4}\ln 4 + \frac{17}{12}\ln 3 - \frac{2}{3}\ln 2 - \frac{1}{12}\right] \\
&= \frac{3}{4}\ln 4 - \frac{15}{12}\ln 3 + \frac{5}{12}\ln 2 \\
&= \frac{1}{12}\left(9\ln 4 - 15\ln 3 + 5\ln 2\right) \\
&= \frac{1}{12}\left(23\ln 2 - 15\ln 3\right)
\end{aligned}
$$

Comments

A long question to answer. One challenge it posed was presenting you with the following integral:

$$\int_a^b \ln y \; dy$$

This I managed to figure out by asking myself, what if I differentiated $y \ln y$? It is a product, obviously, and the product rule would therefore give me $\ln y$ as one of the two terms. Then if I integrated, I might stand a chance of being able to integrate the other term. In fact the other term turned out to be one and so I had reason to celebrate. I am not quite sure whether you are supposed to know this integral but, well, now you know. You might not like my habit of leaving off the limits of integration in these quick calculations, either, but still.

This seems like a good opportunity to discuss integrating by parts, of which the above calculation is, perhaps surprisingly, a good example. For the record I will show you how to derive the standard formula, but I will argue afterwards that you never need to make use of it. For the record, then, we start with the product rule and then integrate:

$$(uv)' = u'v + uv'$$

$$uv = \int u'v + \int uv'$$

This leads to the usual asymmetric formulae, one of which you are supposed to remember:

$$\int u'v = uv - \int uv' \qquad \int uv' = uv - \int u'v$$

However, as I have already pointed out, you can safely forego the formulaic approach. Before I show you a better way to deal with integrating products, which is what integrating by parts really is, let me belittle that approach. Suppose I wanted to find the following integral:

$$\int xe^{4x} \; dx$$

If I were to slavishly apply the formula for integrating by parts, I would have to figure out which of x and e^{4x} equated to u' and v, assuming I was making use of the first of the above formulae that is, then calculate in my head what u and v' were and then, again most probably in my head, figure out whether the product uv' was itself integrable. What I used to do in practice, in order to make my life slightly easier, was to label the two parts of the integrand u' and v. However, I still struggled to hold the rest in my head and although I usually got things out in the end, it always seemed that the calculation was far more difficult than it needed to be.

It all comes down to these needless u's and v's and the laborious substitutions you need to do with them, coupled with the fact that the approach forces you to do the most difficult thing and in fact the only thing that really matters, namely working out whether uv' is itself integrable, most likely in your head and at the end of the sequence of steps.

Figuring out the integral of $\ln y$ made me realise that there is a much better way. All you need to do is to figure out which product when differentiated gives you two products, one of which is the product you are trying to integrate up to a scalar multiple. To make that a little more concrete, in the case I gave you effectively need to figure out the following:

$$(...)' = xe^{4x} + ...$$

Since you know that the derivative of the exponential function is itself and that the derivative of x^2 is $2x$, you might be tempted to try this:

$$(x^2e^{4x})' = (2x)e^{4x} + x^2(4e^{4x})$$

If we divide though by 2 we see that we recover our integrand:

$$\frac{1}{2}(x^2e^{4x})' = xe^{4x} + 2x^2e^{4x}$$

Integrating:

$$\frac{1}{2}x^2e^{4x} = \int xe^{4x}\,dx + \int 2x^2e^{4x}\,dx$$

Now we are stuck, however. We have recovered the integrand, but we have left ourselves with another integral that appears to be even more difficult. Remember the whole point of integrating by parts is to leave yourself with an integral that is easier to solve than the original, not harder!

So we go back to the drawing board, realising that we could choose the product to differentiate as the integrand itself:

$$(xe^{4x})' = e^{4x} + x(4e^{4x})$$

Dividing through by 4 this time:

$$\frac{1}{4}(xe^{4x})' = \frac{1}{4}e^{4x} + xe^{4x}$$

Now we see our integrand again, this time on the right, and we can integrate:

$$\frac{1}{4}xe^{4x} = \frac{1}{4}\int e^{4x}\,dx + \int xe^{4x}\,dx$$

And we are done. We have our required integral on the right and the other integral is easier. We can solve this as the next step or rearrange now and place the integral to find on its own on the left.

I hope I have not obscured the technique here in choosing a slightly fiddlesome integral to solve. I chose it because I feel it is just the kind of integral that the standard approach, with all those u's and v's, makes far more difficult to solve than it should be. So, at the risk of repetition, let me run the logarithm example by you one more time. We are asked to find...

$$\int \ln y\,dy$$

...so we ask, what product do we differentiate in order to get $\ln y$ plus something else:

$$(...)' = \ln y + ...$$

Then it should be clear:

$$(y\ln y)' = \ln y + y(\ln y)'$$

We know that we can differentiate $\ln y$ and in fact we are in luck, because the latter product now just becomes one:

$$(y \ln y)' = \ln y + 1$$

And if we integrate through we are done. No $u's$ and v's in sight!

Now let me return to the remainder of the question. The last two parts required some ingenuity. In the first of these, the numerator inside the logarithm cried out to be factorised and then all you needed was the basic identity $\ln ab = \ln a + \ln b$ and you would hopefully have been on your way. I say that was all you needed, but there was yet more working out to do. One thing I did to save time was to avoid evaluating the integral I_2 from the top. Only the value of the variable b changed and so, as you can see, it was quicker to re-evaluate just the derivative and the limits. Then all that was needed was some careful substitution and to repeat the last few steps.

Question 27

$$
\begin{aligned}
y &= x(x^2 - 2x + \alpha)^{-\frac{1}{2}} \\
\frac{dy}{dx} &= (x^2 - 2x + \alpha)^{-\frac{1}{2}} - \frac{1}{2}x(x^2 - 2x + \alpha)^{-\frac{3}{2}}(2x - 2) \\
0 &= (x^2 - 2x + \alpha)^{-\frac{1}{2}} - x(x-1)(x^2 - 2x + \alpha)^{-\frac{3}{2}} \\
&= (x^2 - 2x + \alpha)^{-\frac{1}{2}}\left[1 - x(x-1)(x^2 - 2x + \alpha)^{-1}\right]
\end{aligned}
$$

Completing the square:

$$
\begin{aligned}
x^2 - 2x + \alpha &= x^2 - 2x + 1 - 1 + \alpha \\
&= (x-1)^2 + \alpha - 1
\end{aligned}
$$

We note that this is strictly positive for all x when $\alpha > 1$, therefore:

$$
\begin{aligned}
1 - x(x-1)(x^2 - 2x + \alpha)^{-1} &= 0 \\
1 &= x(x-1)(x^2 - 2x + \alpha)^{-1} \\
x^2 - 2x + \alpha &= x(x-1) \\
x^2 - 2x + \alpha &= x^2 - x \\
-2x + \alpha &= -x \\
x &= \alpha
\end{aligned}
$$

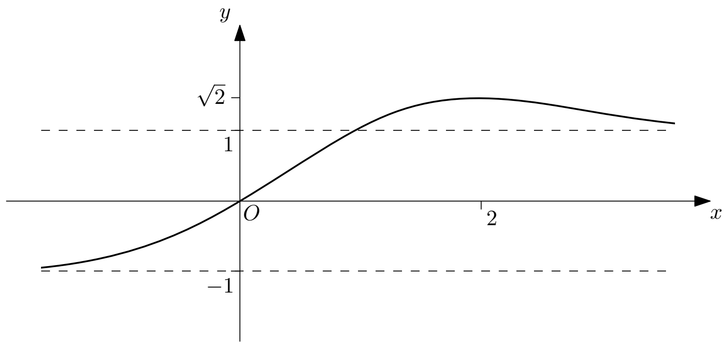
(i) $\alpha = 2$:

The curve passes through the origin, there is a turning point at $x = \alpha = 2$:

$$
y = \frac{2}{\sqrt{(2^2 - 2 \times 2 + 2)}} = \frac{2}{\sqrt{2}} = \sqrt{2}
$$

Also:

$$
\lim_{x \to \pm\infty} \frac{x}{\sqrt{(x^2 - 2x + 2)}} = \lim_{x \to \pm\infty} \frac{x}{\sqrt{(x^2 - \cdots)}} = \lim \frac{x}{|x|} = \pm 1
$$



(i) $\alpha = 1$:

$$y = \frac{x}{\sqrt{(x^2 - 2x - 1)}} = \frac{x}{\sqrt{(x-1)^2}} = \frac{x}{|x-1|}$$

Clearly $y \to \pm 1$ again as $x \to \pm\infty$. The curve also passes through the origin again. To find the turning points, we break the function down into branches:

$$y = \begin{cases} \dfrac{x}{x-1} & x \geqslant 1 \\ \dfrac{x}{1-x} & x < 1 \end{cases}$$

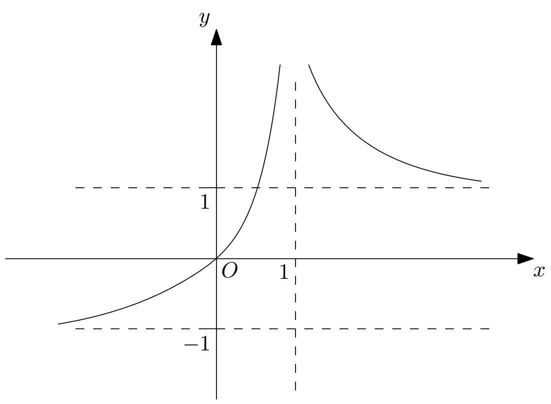
Taking the rightmost branch:

$$\frac{dy}{dx} = \frac{(x-1) - x}{(x-1)^2} = \frac{-1}{(x-1)^2}$$

Hence there are no turning points. Similarly for the leftmost branch:

$$\frac{dy}{dx} = \frac{(1-x) - -x}{(x-1)^2} = \frac{1}{(x-1)^2}$$

As well as horizontal asymptotes at $y = \pm 1$, there is a vertical asymptote at $x = 1$.



Comments

There is not much to say about this question other than I hope it went well for you. The one thing you may not have encountered before is the following identity:

$$|x| = \sqrt{x^2}$$

Obviously you can replace x with any expression.

As usual, I had to separate the curve involving the $|\cdots|$ function into its constituent branches before I could do anything further with it.

Question 28

$$\sum_{k=0}^{n} \sin k\theta = \mathrm{Im}\left[\sum_{k=0}^{n} \cos k\theta + i \sin k\theta\right]$$

$$= \mathrm{Im}\left[\sum_{k=0}^{n} e^{ik\theta}\right]$$

$$= \mathrm{Im}\left[\sum_{k=0}^{n} (e^{i\theta})^k\right]$$

$$\sum_{k=0}^{n} (e^{i\theta})^k = \frac{(e^{i\theta})^{n+1} - 1}{e^{i\theta} - 1}$$

$$= \frac{e^{i(n+1)\theta} - 1}{e^{i\theta} - 1}$$

$$= \frac{\left((e^{i(n+1)\theta} - 1\right) e^{-\frac{1}{2}i\theta}}{(e^{i\theta} - 1)\, e^{-\frac{1}{2}i\theta}}$$

$$= \frac{e^{i(n+\frac{1}{2})\theta} - e^{-\frac{1}{2}i\theta}}{e^{\frac{1}{2}i\theta} - e^{-\frac{1}{2}i\theta}}$$

$$= \frac{e^{i(n+\frac{1}{2})\theta} - e^{-\frac{1}{2}i\theta}}{e^{\frac{1}{2}i\theta} - e^{-\frac{1}{2}i\theta}} . \frac{2i}{2i}$$

$$= \frac{e^{i(n+\frac{1}{2})\theta} - e^{-\frac{1}{2}i\theta}}{2i} \left(\frac{e^{\frac{1}{2}i\theta} - e^{-\frac{1}{2}i\theta}}{2i}\right)^{-1}$$

$$= \frac{1}{2i}\left(e^{i(n+\frac{1}{2})\theta} - e^{-\frac{1}{2}i\theta}\right) \frac{1}{\sin \frac{1}{2}\theta}$$

$$= \frac{1}{2} i \left(e^{-\frac{1}{2}i\theta} - e^{i(n+\frac{1}{2})\theta}\right) \frac{1}{\sin \frac{1}{2}\theta}$$

$$= \frac{1}{2} i \left[\cos -\frac{1}{2}\theta + i \sin -\frac{1}{2}\theta - \cos(n+\frac{1}{2})\theta - i \sin(n+\frac{1}{2})\theta\right] \frac{1}{\sin \frac{1}{2}\theta}$$

$$= \frac{1}{2}\left[i \cos -\frac{1}{2}\theta - \sin -\frac{1}{2}\theta - i \cos(n+\frac{1}{2})\theta + \sin(n+\frac{1}{2})\theta\right] \frac{1}{\sin \frac{1}{2}\theta}$$

$$\therefore \mathrm{Im}\left[\sum_{k=0}^{n} (e^{i\theta})^k\right] = \frac{1}{2}\left[\cos -\frac{1}{2}\theta - \cos(n+\frac{1}{2})\theta\right] \frac{1}{\sin \frac{1}{2}\theta}$$

$$\therefore \sum_{k=0}^{n} \sin k\theta = \frac{\cos -\frac{1}{2}\theta - \cos(n+\frac{1}{2})\theta}{2 \sin \frac{1}{2}\theta} = \frac{\cos \frac{1}{2}\theta - \cos(n+\frac{1}{2})\theta}{2 \sin \frac{1}{2}\theta}$$

(i) Setting $\theta = \dfrac{\pi}{n}$, we only need to show the following:

$$\frac{1}{2}\left[\cos\left(\frac{\pi}{2n}\right) - \cos\left(n+\frac{1}{2}\right)\frac{\pi}{n}\right] = \cos\left(\frac{\pi}{2n}\right)$$

$$\cos\left(\frac{\pi}{2n}\right) - \cos\left(n+\frac{1}{2}\right)\frac{\pi}{n} = 2\cos\left(\frac{\pi}{2n}\right)$$

$$-\cos\left(n+\frac{1}{2}\right)\frac{\pi}{n} = \cos\left(\frac{\pi}{2n}\right)$$

Now:

$$\begin{aligned}
\cos\left(n+\frac{1}{2}\right)\frac{\pi}{n} &= \cos\left(\pi + \frac{\pi}{2n}\right) \\
&= \cos(\pi)\cos\left(\frac{\pi}{2n}\right) - \sin(\pi)\sin\left(\frac{\pi}{2n}\right) \\
&= -\cos\left(\frac{\pi}{2n}\right)
\end{aligned}$$

Hence we are done.

(ii)

$$\begin{aligned}
\cos^2\phi + \sin^2\phi &= 1 \\
\cos^2\phi - \sin^2\phi &= \cos 2\phi \\
1 - 2\sin^2\phi &= \cos 2\phi \\
1 - 2\sin^2\left(\frac{k\theta}{2}\right) &= \cos k\theta \\
k - 2k\sin^2\left(\frac{k\theta}{2}\right) &= k\cos k\theta \\
&= (\sin k\theta)' \\
\sum_{k=1}^{n} k - 2\sum_{k=1}^{n} k\sin^2\left(\frac{k\theta}{2}\right) &= \sum_{k=1}^{n}(\sin k\theta)' \\
&= \left(\sum_{k=1}^{n}\sin k\theta\right)' \\
&= \left(\frac{\cos\frac{1}{2}\theta - \cos(n+\frac{1}{2})\theta}{2\sin\frac{1}{2}\theta}\right)' \\
&= \frac{1}{2}\left(\frac{\cos\frac{1}{2}\theta - \cos(n+\frac{1}{2})\theta}{\sin\frac{1}{2}\theta}\right)' \\
&= \frac{1}{2}\left(\frac{\cos\frac{1}{2}\theta - \cos n\theta\cos\frac{1}{2}\theta + \sin n\theta\sin\frac{1}{2}\theta}{\sin\frac{1}{2}\theta}\right)'
\end{aligned}$$

$$= \frac{1}{2}\left((1-\cos n\theta)\cot\frac{1}{2}\theta + \sin n\theta\right)'$$

$$= \frac{1}{2}\left(\sin n\theta\cot\frac{1}{2}\theta - (1-\cos n\theta)\text{cosec}^2\frac{1}{2}\theta + n\cos n\theta\right)$$

Substituting $\frac{\pi}{n}$ for θ:

$$\dots = \frac{1}{2}\left(\sin\pi\cot\left(\frac{\pi}{2n}\right) - (1-\cos\pi)\text{cosec}^2\left(\frac{\pi}{2n}\right) + n\cos\pi\right)$$

$$= \frac{1}{2}\left(-2\text{cosec}^2\left(\frac{\pi}{2n}\right) - n\right)$$

$$= -\text{cosec}^2\left(\frac{\pi}{2n}\right) - \frac{n}{2}$$

Lastly:

$$\sum_{k=1}^{n} k - 2\sum_{k=1}^{n} k\sin^2\left(\frac{k\theta}{2}\right) = -\text{cosec}^2\left(\frac{\pi}{2n}\right) - \frac{n}{2}$$

$$2\sum_{k=1}^{n} k\sin^2\left(\frac{k\theta}{2}\right) = \sum_{k=1}^{n} k + \text{cosec}^2\left(\frac{\pi}{2n}\right) + \frac{n}{2}$$

$$= \frac{n(n+1)}{2} + \frac{n}{2} + \text{cosec}^2\left(\frac{\pi}{2n}\right)$$

$$= \frac{n^2+2n}{2} + \text{cosec}^2\left(\frac{\pi}{2n}\right)$$

$$= \frac{n^2+2n+1}{2} - \frac{1}{2} + \text{cosec}^2\left(\frac{\pi}{2n}\right)$$

$$= \frac{(n+1)^2}{2} + \frac{1}{2}\left[\text{cosec}^2\left(\frac{\pi}{2n}\right) - 1\right]$$

$$= \frac{(n+1)^2}{2} + \frac{1}{2}\cot^2\left(\frac{\pi}{2n}\right)$$

$$\therefore \sum_{k=1}^{n} k\sin^2\left(\frac{k\theta}{2}\right) = \frac{(n+1)^2}{4} + \frac{1}{4}\cot^2\left(\frac{\pi}{2n}\right)$$

Comments

This question relies heavily one way or another on Euler's formula:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

You should meet a proper proof in your first year of mathematics at university and therefore I do not want to give a hand-wavy one here. It is difficult to underestimate this formula's utility. For a start, you can generalise problems involving trigonometric functions to ones involving the exponential function, and in doing so you will stand a far better chance of solving them. These two identities look pretty innocuous, but they are fundamental:

$$\cos\theta = \mathrm{Re}\big[e^{i\theta}\big] \quad \sin\theta = \mathrm{Im}\big[e^{i\theta}\big]$$

They are trivial to prove using Euler's formula. I have used the second, and in fact inadvertently its proof, in the first two lines of my answer.

Euler's formula is aided and abetted by some simple observations about indexes applied to the exponential function. For example:

$$\left(e^{i\theta}\right)^n = e^{in\theta}$$

If you are not yet a believer, consider the following:

$$\begin{aligned}
\cos(A+B) + i\sin(A+B) &= e^{i(A+B)} \\
&= e^{iA+iB} \\
&= e^{iA}e^{iB} \\
&= (\cos A + i\sin A)(\cos B + i\sin B) \\
&= \cos A\cos B + i\cos A\sin B + i\sin A\cos B + i^2\sin A\sin B) \\
&= [\cos A\cos B - \sin A\sin B] + i\,[\cos A\sin B + \sin A\cos B]
\end{aligned}$$

Equating real and imaginary parts gives the familiar identities:

$$\begin{aligned}
\sin(A+B) &= \sin A\cos B + \cos A\sin B \\
\cos(A+B) &= \cos A\cos B - \sin A\sin B
\end{aligned}$$

We can also express the trigonometric functions in terms of the exponential function. The following identities are easy to check:

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Note the slight asymmetry in that the denominator of the second identity is imaginary. All of this was put to good use in the first part of the question. Even by my standards I was cautious, but I wanted to show every application of all of the above.

The second part could be done relatively quickly if you took the opportunity to not show all the working out. I think this is fine provided you tell the examiner exactly where you are at.

The last part I think is too long. The first time I attempted it, I missed the fact that the right hand side of the first equation could be simplified to get rid of the fraction. I remember contemplating this with my student and deciding that it would likely only make things more complicated. Looking back on it, it seems silly that we did not at least try. Leaving ourselves with having to apply the quotient rule meant reams of working out for both of us and inevitably mistakes crept in. So I think a hint would have been good, but of course I would say that. Even doing things the easy way, you can see that I have had to massage things a little bit at the end in order to get it right. There is no half in front of the cosec^2 term in the fourth from last line but the following line assumes that there is. I tried tracking back to find my mistake but it has eluded me, hence the ruse. If you happen to find my mistake, please let me know!

Question 29

(i)

$$\left(x-ak^{-1}\right)(x-a)(x-ka)=x^3-(ak^{-1}+a+ka)x^2+(ak^{-1}a+ak^{-1}ka+aka)x-ak^{-1}aka$$

$$\left.\begin{aligned}-p&=(ak^{-1}+a+ak)=-a(k^{-1}+1+k)\\ q&=(a^2k^{-1}+a^2+a^2k)=a^2(k^{-1}+1+k)\\ -r&=-a^3\end{aligned}\right\}$$

$$\frac{q}{-p}=\frac{a^2}{-a}\qquad\therefore\frac{q}{p}=a$$

$$q^3-rp^3=a^6(k^{-1}+1+k)^3-a^3a^3(k^{-1}+1+k)^3=0$$

(ii)

$$\begin{aligned}0&=x^3-px^2+qx-\frac{q^3}{p^3}\\ &=p^3x^3-p^4x^2+p^3qx-q^3\\ &=(px)^3-p^2(px)^2+p^2q(px)-q^3\end{aligned}$$

Substituting y for px:

$$y^3-p^2y^2+p^2qy-q^3=0$$

By inspection $y=q$ is a root of this equation, hence $x=q/p$ is a root of the original equation.

Dividing $(y-q)$ into $y^3-p^2y^2+p^2qy-q^3$ gives the following:

$$y^2+(q-p^2)y+q^2=0$$

Returning back to x:

$$(px)^2+(q-p^2)px+q^2=0$$

$$p^2x^2+(q-p^2)px+q^2=0$$

$$x^2+\frac{q-p^2}{p}x+\frac{q^2}{p^2}=0$$

Therefore the product of the remaining two roots is $(q/p)^2$.

Suppose the three roots are α, β and γ. Without loss of generality we can write:

$$\alpha=q/p,\quad\beta\gamma=(q/p)^2$$

We note that β and γ are non-zero because their product is non-zero. We also note $\alpha^2=\beta\gamma$ and dividing through by β^2 we get:

$$\left(\frac{\alpha}{\beta}\right)^2=\frac{\gamma}{\beta}$$

Since the ratio of the third and first root is the square of the ratio of the second and first root, they form a geometric progression. To argue more formally, we can choose a constant k such that:

$$\begin{aligned}\left(\frac{\alpha}{\beta}\right)^2 &= k^2\\ \alpha^2 &= \beta^2 k^2\\ \alpha &= \pm\beta k\end{aligned}$$

Therefore the roots are β, $\pm k\beta$, $k^2\beta$, which is a geometric progression.

(iii) To begin with we note that, from part (i), if we assume the roots are in geometric progression, then one root is p/q and additionally $q^3 - rp^3 = 0$. Conversely we note that, from part (ii), if we assume that r $= q^3/p$, then the roots are in geometric progression. We follow the same pattern to prove the similar, required result about arithmetic progressions.

We assume that the roots are in arithmetic progression, that is, they can be written $a-k$, a, $a+k$. Then:

$$\begin{aligned}x^3 - px^2 + qx - r &= (x-a+k)(x-a)(x-a-k)\\ &= x^3 - (a+k+a+a-k)x^2 + (a(a+k) + (a+k)(a-k) + a(a-k)) + a(a-k)(a+k)\\ &= x^3 - 3ax^2 + (a^2 + ak + a^2 - k^2 + a^2 - ak)x - a(a^2-k^2)\\ &= x^3 - 3ax^2 + (3a^2 - k^2)x - (a^3 + ak^2)\end{aligned}$$

Equating coefficients:

$$\left.\begin{aligned}-p &= -3a\\ q &= 3a^2 - k^2\\ -r &= -a^3 + ak^2\end{aligned}\right\}$$

This gives one root as $p = 3a$ and since we now know we can write a in terms of p, we need to eliminate k from the second and third equations in order to get an equation involving solely p, q and r:

$$\left.\begin{aligned}3a^2 - k^2 &= q\\ a^3 - ak^2 &= r\end{aligned}\right\}$$

$$\left.\begin{aligned}3a^3 - ak^2 &= aq\\ a^3 - ak^2 &= r\end{aligned}\right\}$$

$$\begin{aligned}2a^3 &= qa - r\\ \therefore 2\left(\frac{p}{3}\right)^3 &= \frac{pq}{3} - r\end{aligned}$$

Conversely, suppose we set r to be the following:

$$\frac{pq}{3} - 2\left(\frac{p}{3}\right)^3$$

Substituting into the original equation, we get:

$$x^3 - px^2 + qx - \frac{pq}{3} + 2\left(\frac{p}{3}\right)^3 = 0$$

We can see that $p/3$ is a root:

$$\begin{aligned}
\left(\frac{p}{3}\right)^3 - p\left(\frac{p}{3}\right)^2 + q\left(\frac{p}{3}\right) - \frac{pq}{3} + \left(\frac{p}{3}\right)^3 &= 0 \\
\frac{p^3}{27} - \frac{p^3}{9} + \frac{pq}{3} - \frac{pq}{3} + \frac{2p^3}{27} &= 0 \\
p^3\left(\frac{1}{27} - \frac{1}{9} + \frac{2}{27}\right) &= 0 \\
0 &= 0
\end{aligned}$$

The sum of the remaining two roots is therefore $2p/3$. Their mean is $p/3$ and they can therefore be written:

$$\beta = \frac{p}{3} - k, \quad \gamma = \frac{p}{3} + k$$

That is, they are in arithmetic progression.

Comments

This question is well worth its three stars. The first part is not so bad if you see that all you have to do is substitute into the equation $q^3 - rp^3$ rather than attempt to derive it in any way. After that, things get interesting.

The second part is more difficult in certain places. There was no need to do the algebraic long division in order to arrive at a quadratic involving y, as I did. Simply noting that r was product of the three roots and then dividing by the one root you had found would have gotten you there quickly. There were other pitfalls. Did you realise that you were not required to solve the remaining two simultaneous equations in order to arrive at explicit formulae for the remaining two roots? I did, but on my first pass I could think of nothing else to do. So I tried, and regretted it. Fortunately the penny dropped when I gave up on that route and considered the problem a second time. You can see that I gave two explanations as to why the roots were in geometric progression, when obviously one would have sufficed.

The trick with the third part was to realise that the examiners had given you the template in the first two. I even went so far as to copy out the argument so that I had something to work from. Getting the equation involving p, q and r was really just a question of following the exact same argument from the first part, but still a little ingenuity was required to get right over the line. Then you had to realise that you needed to take the formula you had derived and go back the other way. There was really nothing else that could be done, you certainly could not be expected to dream up anything else, but I nonetheless started off down that track wondering whether I had missed something obvious or done something stupid. And to really get started in this direction, you had to realise that $p/3$ was a root of the cubic equation. I guessed because I saw that the $pq/3$ term would cancel out the linear term, but I distinctly remember the last time I attempted this question with a student, it was another one of those how was I supposed to know moments. After that, you again had to avoid the pitfall of trying to find the roots explicitly by solving the simultaneous equations. There is a nice finish in that one of the equations governing the remaining two roots gives their mean. So it is easy in that sense, but only if you saw it.

A clever question. With all the thinking involved, I cannot say that I would have succeeded under examination conditions. If you got all the way through in anything like a reasonable amount of time, I think you did extremely well.

Question 30

(i)

$$f(x) = (1 + x^2)e^x - k$$

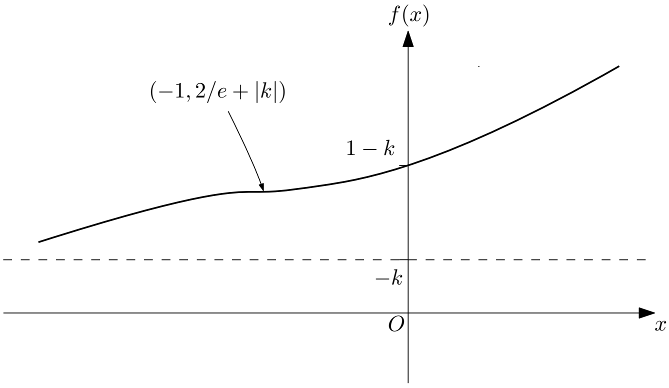
$$f'(x) = (1 + x^2)e^x + 2xe^x$$

$$= (1 + 2x + x^2)e^x$$

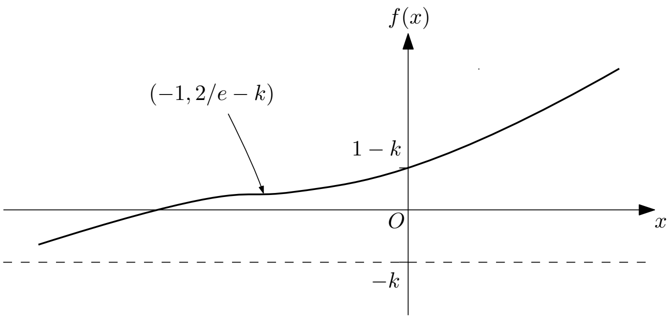
$$= (1 + x)^2 e^x$$

For all x, $(1 + x)^2 \geqslant 0$ and $e^x > 0$ hence $(1 + x)^2 e^x \geqslant 0$, as required.

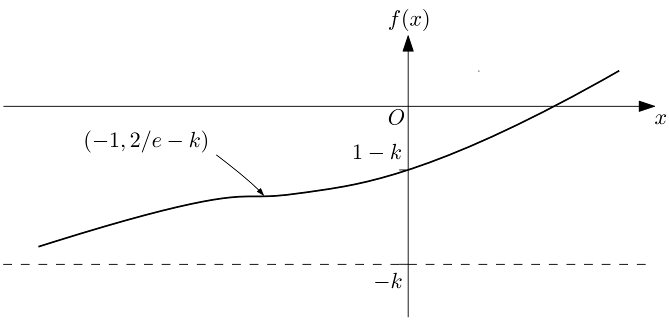
For $k < 0$ we write $f(x) = (1 + x^2)e^x + |k|$. Setting $f'(x) = 0$ gives $x = -1$ so there is one turning point at $(-1, 2/e + |k|)$. The curve also passes through the point $(0, 1 + |k|)$. When $x \to \infty$, $f(x) \to \infty$ and $f'(x) \to \infty$, whilst when $x \to -\infty$, $f(x) \to |k|$ and $f'(x) \to 0$:



For $0 < k < 1$ the graph is translated downwards so that there is an intersection with the x-axis. Otherwise the shape stays the same:



For $1 < k$ the graph is translated downwards again so that the intersection with the x-axis is for positive x:



As the graphs show, since the curve never drops below the asymptote at $-k$, we require $-k < 0$ in order for there to be a root, that is $k > 0$.

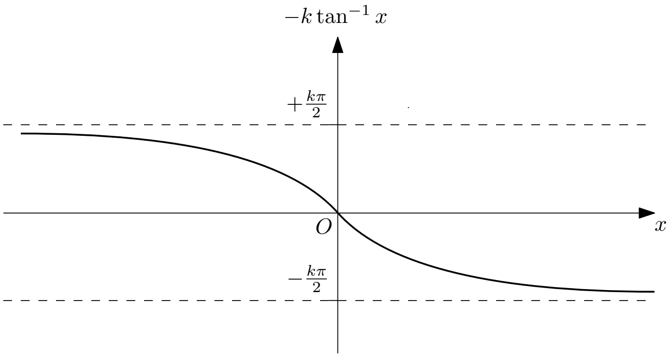
(ii)

$$
\begin{aligned}
g(x) &= (e^x - 1) - k\tan^{-1} x \\
g'(x) &= e^x - \frac{k}{1+x^2} \\
&= \frac{(1+x^2)e^x - k}{1+x^2} \\
&= \frac{f(x)}{1+x^2}
\end{aligned}
$$

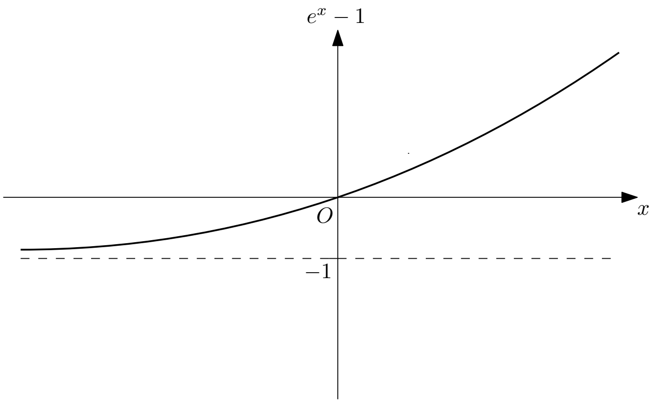
Therefore the turning points of $g(x)$ are given by the roots of $f(x)$.

We know that $f(x)$ has no real roots if $k \leqslant 0$, so we consider this case first. We note that $g(x)$ passes through the origin and since it has no turning points, the sole root is at $x = 0$.

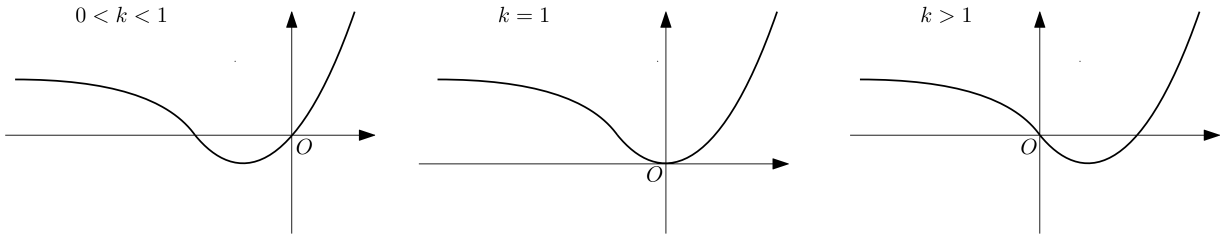
Here is the graph of $-k\tan^{-1} x$:



Here is the graph of $e^x - 1$:



We can see that for all $k > 0$, the second curve eventually dominates the first curve. This results in a single turning point, as shown earlier. We break down $k > 0$ into the three cases $0 < k < 1$, $k = 1$ and $k > 1$. These give turning points at $x < 0$, $x = 0$ and $x > 0$ respectively:



The function always has a root at $x = 0$. From the graphs we can see that for $k > 1$, there is an additional root at $x > 0$.

For the first case, we have to further break it down depending on the value of k. We note that there is an asymptote at $-1 + k\pi/2$. We see that when this is positive, as in the graph shown, there is an additional root at $x < 0$. When it is zero or negative, however, the curve does not climb above the x-axis for negative x and therefore there are no more roots.

Comments

This is a difficult question, not because any single part of it is intrinsically difficult, but because you have to be extremely careful in your reasoning in the second part. I have to admit that I did not make a good job of it the first time around, messing up the various cases in the second part. As I come to type it up, I think I have them right. In this sense I think I would call this question one of those asymptotic questions, no pun intended, in the sense that the reasoning you have to employ becomes clearer each time you attempt it.

On a happier note, the examiners very kindly gave you the derivative of $\tan^{-1} x$. Not only would this have saved you the trouble of working it out at the time, but it also served as a very broad hint as to how to tackle the second part. They could hardly have been any more helpful had they simply told you to differentiate the equation, in fact. Anyway, here is the proof. It uses the same technique as the proof of the derivative of $\ln x$, that is when faced with finding the derivative of dy/dx, you find dx/dy and go from there. Here it is:

$$\begin{aligned}
y &= \tan^{-1} x \\
x &= \tan y \\
\frac{dx}{dy} &= \sec^2 y \\
&= 1 + \tan^2 y \\
&= 1 + x^2 \\
\therefore \frac{dy}{dx} &= \frac{1}{1+x^2}
\end{aligned}$$

Question 31

(i)

$$\begin{aligned}
b_n &= \frac{1}{n+1} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} + \cdots \\
&= \frac{1}{n+1}\left[\frac{1}{n+1} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} + \cdots\right] \\
&= \frac{1}{n+1}(1+b_n) \\
(n+1)b_n &= 1 + b_n \\
nb_n + b_n &= 1 + b_n \\
nb_n &= 1 \\
b_n &= \frac{1}{n}
\end{aligned}$$

(ii) $0 < a_n$ follows from each term being strictly positive. $a_n < b_n$ follows from each term of a_n being less than the corresponding term in b_n.

(iii)

$$\begin{aligned}
e &= 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots \\
n!e &= n! + \frac{n!}{1!} + \frac{n!}{2!} + \frac{n!}{3!} + \cdots \frac{n!}{n!} + \frac{n!}{(n+1)!} + \cdots \\
&= n! + \frac{n!}{1!} + \frac{n!}{2!} + \cdots + \frac{n!}{(n-1)!} + 1 + a_n
\end{aligned}$$

We note that since $1!$, $2!$, $\ldots n!$ all divide $n!$, all the terms bar the a_n term in the sequence above are integers. Furthermore, since $a_n < 1/n < 1$, a_n is the fractional part of $n!e$, as required.

(iv) Suppose that e is rational:

$$\begin{aligned}
e &= \frac{p}{q} \\
a_n &= n!e - [n!e] \\
a_n &= n!\frac{p}{q} - \left[n!\frac{p}{q}\right] \\
\therefore n!\frac{p}{q} &> \left[n!\frac{p}{q}\right]
\end{aligned}$$

This inequality holds for all n, so we choose $n = q$:

$$q!\frac{p}{q} > \left[q!\frac{p}{q}\right]$$

$$(q-1)!p > [(q-1)!p]$$

However $[(q-1)!p] = (q-1)!p$, which leaves:

$$(q-1)!p > (q-1)!p$$

A contraction. Hence e cannot be rational.

Comments

I liked this question, quite possibly because it gave me no trouble. I am not sure that my students would agree with my assessment, however. I contend that the only bits that are not entirely straightforward can found in the third part. You have to see where a_n is involved in the expansion of $n!e$, and also that each of terms before the a_n terms are in fact integers.

When I came to type this up, I found a proof that the derivative of e^x is e^x. I presume I wrote it down to show a student, as this question has nothing to do with differentiation. So just in case you have never seen it, here it is:

$$\begin{aligned}
e^x &= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \\
(e^x)' &= 0 + \frac{1}{1!} + \frac{2x}{2!} + \frac{3x^2}{3!} + \cdots \\
&= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \\
&= e^x
\end{aligned}$$

Question 32

(i)

$$\begin{aligned}\log_{10} 5 &= \log_{10} 10/2 \\ &= \log_{10} 10 - \log_{10} 2 \\ &= 1 - 0.3010 \quad \text{(4dp)} \\ &= 0.699 \quad \text{(3dp)}\end{aligned}$$

$$\begin{aligned}\log_{10} 6 &= \log_{10} 2 * 3 \\ &= \log_{10} 2 + \log_{10} 3 \\ &= 0.3010 + 0.4771 \quad \text{(4dp)} \\ &= 0.778 \quad \text{(3dp)}\end{aligned}$$

$$5 \times 10^{47} < 3^{100} < 6 \times 10^{47}$$
$$\log 5 + 47 \log 10 < 100 \log 3 < \log 6 + 47 \log 10$$
$$\log 5 + 47 < 100 \log 3 < \log 6 + 47$$

Dealing with the left hand inequality, since $\log 5 = 0.699$ (3dp):

$$\begin{aligned}\log 5 &< 0.6995 \\ \log 5 + 47 &< 0.6995 + 47 \\ &= 47.6995 \\ &< 47.7 \\ &= 100 \times 0.477 \\ &< 100 \log 3\end{aligned}$$

And the right hand inequality, since $\log 6 = 0.778$ (3dp):

$$\begin{aligned}\log 6 &\geqslant 0.7775 \\ \log 6 + 47 &\geqslant 0.7775 + 47 \\ &= 47.7775 \\ &> 47.777 \\ &= 100 \times 0.4777 \\ &> 100 \log 3\end{aligned}$$

Therefore the first digit of 3^{100} must be 5.

(ii) We need to choose n and m such that:

$$n \times 10^m < 2^{1000} < (n+1) \times 10^m$$

In which case the first digit of 2^{1000} will be n. Taking logarithms as before:

$$m + \log n < 1000 \log 2 < m + \log(n+1)$$

We have:

$$0.30102 < \log 2 < 0.30103$$

$$301.02 < 1000 \log 2 < 301.03$$

$$301 + 0.02 < 1000 \log 2 < 301 + 0.03$$

So we choose $m = 301$. We then have to find an n such that $\log n < 0.02$ while at the same time $0.03 < \log(n+1)$. Clearly $n = 1$ since $\log 1 = 0 < 0.02$ and $\log 2 \simeq 0.031 > 0.03$.

Similarly:

$$n \times 10^m < 2^{10\ 000} < (n+1) \times 10^m$$

$$m + \log n < 10\ 000 \log 2 < m + \log(n+1)$$

So again:

$$0.301029996 < \log 2 < 0.301029997$$

$$3010.29996 < 10\ 000 \log 2 < 3010.29997$$

$$3010 + 0.29996 < 10\ 000 \log 2 < 3010 + 0.29997$$

So we choose $m = 3010$. We then have to find an n such that $\log n < 0.29996$ while at the same time $0.29997 < \log(n+1)$. By the same arguments $n = 1$.

Finally:

$$n \times 10^m < 2^{100\ 000} < (n+1) \times 10^m$$

$$m + \log n < 100\ 000 \log 2 < m + \log(n+1)$$

So again:

$$0.301029996 < \log 2 < 0.301029997$$

$$30102.9996 < 100\ 000 \log 2 < 30102.9997$$

$$30102 + 0.9996 < 100\ 000 \log 2 < 30102 + 0.9997$$

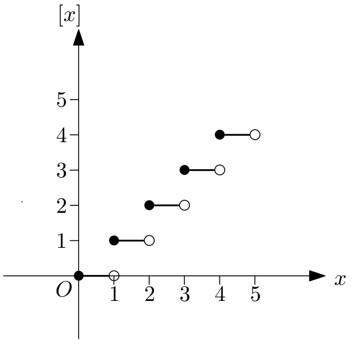
So we choose $m = 30102$. We then have to find an n such that $\log n < 0.9996$ while at the same time $0.9997 < \log(n+1)$. So we choose $n = 9$ since $\log 10 = 1$ is barely more than 0.9996.

Comments

The only thing I would note about this question is that you can go into automaton mode for the last two cases of the second part. Only the choice of n and how many decimal places of $\log 2$ to take should require any thought. Actually, to type it up I just copied and pasted with no thought at all. In the examination you do not have this option, obviously, but the approach should be the same.

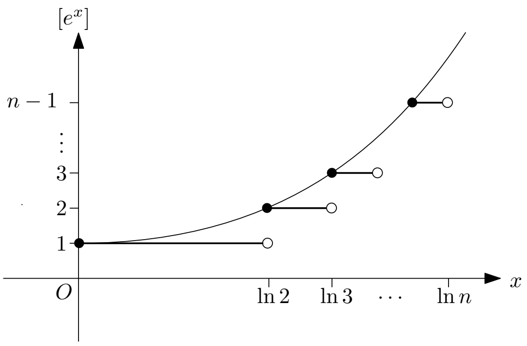
Question 33

(i)



$$\int_0^5 [x]dx = 1 + 2 + 3 + 4 = 10$$

(ii)



$$\begin{aligned}
\int_0^{\ln n} [e^x]\, dx &= 1\ln 2 + 2(\ln 3 - \ln 2) + \cdots + (n-1)(\ln n - \ln(n-1)) \\
&= [1\ln 2 - 2\ln 2] + [2\ln 3 - 3\ln 3] + \cdots + [(n-2)\ln(n-1) - (n-1)\ln(n-1)] + (n-1)\ln n \\
&= -ln2 - \ln 3 - \cdots - \ln(n-1) + (n-1)\ln n \\
&= -\ln 2 - \ln 3 - \cdots - \ln(n-1) - \ln n + \ln n + (n-1)\ln n \\
&= -\ln(n!) + n\ln n
\end{aligned}$$

Lastly:

$$[e^x] \leqslant e^x$$

$$\int_0^{\ln n} [e^x]\, dx \leqslant \int_0^{\ln n} e^x dx$$

$$n\ln n - \ln(n!) \leqslant n - 1$$

$$n\ln n - n + 1 \leqslant \ln(n!)$$

$$e^{n \ln n - n + 1} \leqslant e^{\ln(n!)}$$

$$e^{n(\ln n - 1) + 1} \leqslant n!$$

Simplifying the left hand side:

$$\begin{aligned}
e^{n(\ln n-1)+1} &= e\ e^{n(\ln n-1)} \\
&= e\ \left(e^{\ln n-1}\right)^n \\
&= e\ \left(ne^{-1}\right)^n \\
&= e\ n^n \left(e^{-1}\right)^n \\
&= e\ n^n e^{-n} \\
&= n^n e^{1-n}
\end{aligned}$$

Comments

I have wondered before about the rate of increase of the factorial function. It seems reasonably obvious that it grows more quickly than the exponential function, in fact a lot more quickly. This question provides a lower bound.

An upper bound can be found by taking rectangles in the graph with base $[\ln(n-1), \ln n]$ but with height n rather than $n-1$. This gives the following inequality:

$$\int_0^{\ln n} e^x dx \leqslant \int_0^{\ln n} [e^x] + 1 \ dx$$

This leads to the following:

$$n - 1 \leqslant (n+1)\ln n - \ln(n!)$$

Proceeding exactly as before leads to:

$$n! \leqslant n^{n+1}e^{-n}$$

Therefore:

$$n^n e^{1-n} \leqslant n! \leqslant n^{n+1}e^{-n}$$

Do these bounds provide much insight? I would say no, because of the negative power $-n$ of e. With some rearranging, however, they at least provide the following bounds for the function n^n:

$$(n-1)!\, e^n \leqslant n^n \leqslant n!\, e^n$$

I like these bounds slightly more than the previous ones.

Question 34

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\begin{aligned}\tan 2\theta &= \frac{2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta}\\ &= \frac{2\tan\theta}{1-\tan^2\theta}\end{aligned}$$

Setting $\theta = \pi/8$ and $t = \tan\theta$:

$$\begin{aligned}\frac{2t}{1-t^2} &= 1\\ 2t &= 1-t^2\\ t^2+2t-1 &= 0\\ t^2+2t+1-1-1 &= 0\\ (t+1)^2 &= 2\\ t &= -1\pm\sqrt{2}\end{aligned}$$

Since t is positive, $t = \sqrt{2}-1$ as required.

$$x = \sin 4t \qquad\qquad \sin 4t = \pm 1$$

$$dx = 4\cos 4t\;dt \qquad\qquad 4t = \pm\frac{\pi}{2}$$

$$t = \pm\frac{\pi}{8}$$

$$I = \int_{-\pi/8}^{\pi/8} \frac{1}{\sqrt{(1+\sin 4t)}+\sqrt{(1-\sin 4t)}+2} 4\cos 4t\;dt$$

The integrand is even because $\cos 4(-t) = \cos -4t = \cos 4t$ and since $\sin -4t = -\sin 4t$, the contents of the two square roots in the denominator simply swap places, leaving it the same. Thus:

$$\begin{aligned}I &= 2\int_0^{\pi/8} \frac{1}{\sqrt{(1+\sin 4t)}+\sqrt{(1-\sin 4t)}+2} 4\cos 4t\;dt\\ &= \int_0^{\pi/8} \frac{8\cos 4t}{\sqrt{(1+\sin 4t)}+\sqrt{(1-\sin 4t)}+2} dt\end{aligned}$$

In order to show the required identity it only remains to prove the following:

$$\frac{2\cos 4t}{\cos^2 t} = \frac{8\cos 4t}{\sqrt{(1+\sin 4t)}+\sqrt{(1-\sin 4t)}+2}$$

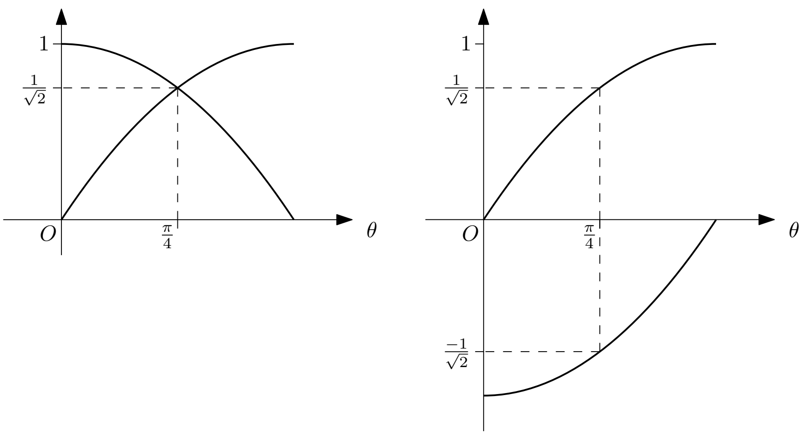
$$\frac{4}{\sqrt{(1+\sin 4t)}+\sqrt{(1-\sin 4t)}+2} = \frac{1}{\cos^2 t}$$

$$\begin{aligned}
4\cos^2 t &= \sqrt{(1+\sin 4t)}+\sqrt{(1-\sin 4t)}+2 \\
&= \sqrt{(1+2\sin 2t\cos 2t)}+\sqrt{(1-2\sin 2t\cos 2t)}+2 \\
&= \sqrt{(\sin^2 2t+2\sin 2t\cos 2t+\cos^2 2t)}+\sqrt{(\sin^2 2t-2\sin 2t\cos 2t+\cos^2 2t)}+2 \\
&= \sqrt{(\sin 2t+\cos 2t)^2}+\sqrt{(\sin 2t-\cos 2t)^2}+2 \\
&= |\sin 2t+\cos 2t|+|\sin 2t-\cos 2t|+2 \\
&= 2cos2t+2
\end{aligned}$$

And since $\cos 2t = 2\cos^2 t - 1$, we are done, if we can prove the following on the interval $[0, \pi/8]$:

$$|\sin 2t+\cos 2t|+|\sin 2t-\cos 2t| = 2\cos 2t$$

We argue graphically. The graphs below show curves for $\sin\theta$ versus $\cos\theta$ and $\sin\theta$ versus $-\cos\theta$ on the interval $[0, \pi/4]$:



We can see that both $\sin\theta$ and $\cos\theta$ are positive on required interval, hence:

$$|\sin\theta+\cos\theta| = \sin\theta+\cos\theta$$

We can also see that $\cos\theta \geqslant \sin\theta$ on the given interval, hence:

$$|\sin\theta-\cos\theta| = \cos\theta-\sin\theta$$

$$\therefore |\sin\theta+\cos\theta|+|\sin\theta-\cos\theta| = \sin\theta+\cos\theta+\cos\theta-\sin\theta = 2\cos\theta \quad 0 \leqslant \theta \leqslant \frac{\pi}{4}$$

Setting $\theta = 2t$ gives the required result.

$$\begin{aligned}
I &= \int_0^{\pi/8} \frac{2\cos 4t}{\cos^2 t}\, dt \\
&= \int_0^{\pi/8} \frac{4\cos^2 2t - 2}{\cos^2 t}\, dt
\end{aligned}$$

$$= \int_0^{\pi/8} \frac{4(2\cos^2 t - 1)^2 - 2}{\cos^2 t}\ dt$$

$$= \int_0^{\pi/8} \frac{16\cos^4 2t - 16\cos^2 t + 4 - 2}{\cos^2 t}\ dt$$

$$= \int_0^{\pi/8} 16\cos^2 t - 16 + 2\mathrm{sec}^2 t\ dt$$

$$= \int_0^{\pi/8} 8\cos 2t - 8 + 2\mathrm{sec}^2 t\ dt$$

$$= [4\sin 2t]_0^{\pi/8} - [8t]_0^{\pi/8} + [2\tan t]_0^{\pi/8}$$

$$= 4\sin\frac{\pi}{4} - \pi + 2(\sqrt{2} - 1)$$

$$= 4\frac{4}{\sqrt{2}} - \pi + 2\sqrt{2} - 2$$

$$= 4\sqrt{2} - \pi - 2$$

Comments

This is a long question but a fair one. For the second part you are given the substitution, otherwise the question would be pretty impossible. Note that I did not carry the derivation straight down from start to finish. I stated that there was an identity to prove and then essentially worked backwards. I think this is absolutely fine, on the basis that if you were to slavishly rewrite the whole lot and enclose it in the integral, no more understanding would really have been communicated.

There are a few steps in that derivation that deserve mention. You may ask, as more than one student has, how are you supposed to know that the nasty looking square roots could be simplified in that way? All I can say is that when faced with an expression involving trigonometric functions of t on the one side and an expression involving trigonometric functions of $4t$ on the other, you have little choice but to use trigonometric identities to go from t to $4t$, or vice versa. I guessed the right way to go because of the square roots. I did not foresee that I would need the identity $1 = \sin^2 2t + \cos^2 2t$, by the way, nor did I have any idea beforehand that $|\sin 2t - \cos 2t| = \cos 2t - \sin 2t$ on the given interval. I mention these steps because they also brought about comments of befuddlement from students.

Question 35

(i)

$$\begin{aligned}
f(x) &= x^6(x^2+1)^{-4} \\
f'(x) &= 6x^5(x^2+1)^{-4} + -4x^6(x^2+1)^{-5}.2x \\
&= 6x^5(x^2+1)^{-4} + -8x^7(x^2+1)^{-5} \\
&= 2x^5(x^2+1)^{-5}\left[3(x^2+1) - 4x^2\right] \\
&= 2x^5(x^2+1)^{-5}(3-x^2)
\end{aligned}$$

Setting $f'(x) = 0$ gives $x = 0$ or $x = \pm\sqrt{3}$. Since $\sqrt{3} > 1$, those two turning points lie outside of the interval $[0, 1]$. Clearly $f(x) \geqslant 0$ on the same interval and $f(0) = 0$ therefore $f(x)$ must be increasing on $[0, 1]$. Hence its greatest value is at $x = 1$:

$$f(1) = \frac{1^6}{(1^2+1)^4} = \frac{1}{16}$$

By the same reasoning the smallest value must be at $x = 0$ and is 0.

(ii) Differentiating:

$$\begin{aligned}
\frac{d}{dx}\left(\frac{Ax^5+Bx^3+Cx}{(x^2+1)^3}\right) &= (5Ax^4+3Bx^2+C)(x^2+1)^{-3} + (Ax^5+Bx^3+Cx).-3(x^2+1)^{-4}.2x \\
&= (5Ax^4+3Bx^2+C)(x^2+1)(x^2+1)^{-4} - 6x(Ax^5+Bx^3+Cx)(x^2+1)^{-4}
\end{aligned}$$

Equating just the numerators:

$$\begin{aligned}
1 &= (5Ax^4+3Bx^2+C)(x^2+1) - 6x(Ax^5+Bx^3+Cx) + Dx^6 \\
&= 5Ax^6+3Bx^4+Cx^2+5Ax^4+3Bx^2+C-6Ax^6-6Bx^4-6Cx^2+Dx^6 \\
&= (5A-6A+D)x^6 + (3B+5A-6B)x^4 + (C+3B-6C)x^2 + C
\end{aligned}$$

Equating coefficients:

$$\left.\begin{aligned}
-A+D &= 0 \\
5A-3B &= 0 \\
3B-5C &= 0 \\
C &= 1
\end{aligned}\right\}$$

This gives $C = 1$, $B = 5/3$, $A = 1$ and $D = 1$.

(iii) From part (i) we see immediately that:

$$0 < \int_0^1 \frac{x^6}{(x^2+1)^4}dx < \frac{1}{16}$$

Given the equivalence in part (ii) we have:

$$0 < \int_0^1 \frac{1}{(x^2+1)^4}dx - \left[\frac{x^5 + {}^5\!/\!_3x^3 + x}{(x^2+1)^3}\right]_0^1 < \frac{1}{16}$$

Evaluating:

$$\left[\frac{x^5 + {}^5\!/\!_3x^3 + x}{(x^2+1)^3}\right]_0^1 = \frac{1 + {}^5\!/\!_3 + 1}{(1^2+1)^3} = \frac{{}^{11}\!/\!_3}{8} = \frac{11}{24}$$

Replacing the strict inequalities $<$ with $\leqslant$, which we are free to do, gives the required result.

Comments

There is not much I feel I can say about this question. The first part required a little common sense, perhaps. In the last part, from considering the integral it should be clear that the strict inequalities hold. The question asks for the weaker result, however, so I added a sentence to qualify. The only other thing I can think of that might help is that I used the product rule rather than the quotient rule and tried to avoid as much writing as possible. The examiner will be fine with the latter approach as long as you explain clearly what you are doing.

Question 36

Since $e^x > 0$ for all x we can write:

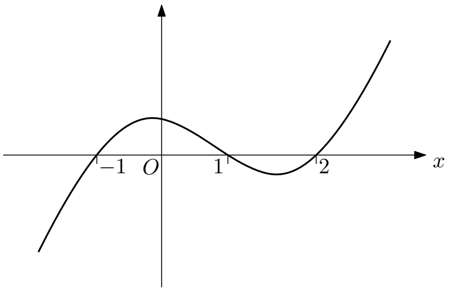
$$|xe^x| = \begin{cases} xe^x = x \geqslant 0 \\ -xe^x = x < 0 \end{cases}$$

$$\begin{aligned}
\therefore \int_{-1}^{1} |xe^x| dx &= \int_{-1}^{0} -xe^x \, dx + \int_{0}^{1} xe^x \, dx \\
&= \int_{0}^{1} xe^x \, dx - \int_{-1}^{0} xe^x \, dx \\
&= \int_{0}^{1} xe^x \, dx + \int_{0}^{-1} xe^x \, dx
\end{aligned}$$

Integrating by parts:

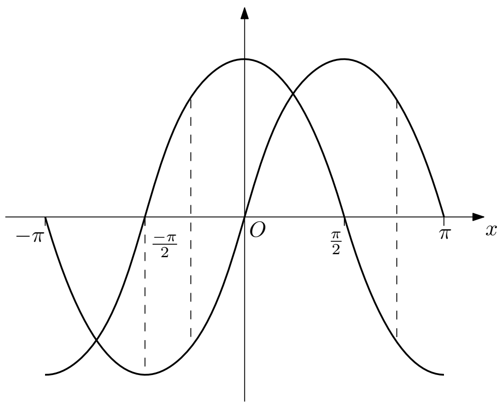
$$\begin{aligned}
\frac{d}{dx}(xe^x) &= xe^x + e^x \\
[xe^x]_a^b &= \int_a^b xe^x \, dx + \int_a^b e^x \, dx \\
\int_a^b xe^x \, dx &= [xe^x]_a^b - \int_a^b e^x \, dx \\
&= [xe^x]_a^b - [e^x]_a^b \\
&= [(x-1)e^x]_a^b \\
\therefore \int_{-1}^{1} |xe^x| dx &= [(x-1)e^x]_0^1 + [(x-1)e^x]_0^{-1} \\
&= \left[(1-1)e^1 - (0-1)e^0\right] + \left[(-1-1)e^{-1} - (0-1)e^0\right] \\
&= e^0 - 2e^{-1} + e^0 \\
&= 2(1 - {}^1\!/e)
\end{aligned}$$

(i) By inspection we can see that $x = 1$ is a root. Dividing leaves the quadratic $x^2 - x - 2$ and again by inspection this factorises into $(x+1)(x-2)$. Therefore the graph is:



$$
\begin{aligned}
\therefore \int_0^4 |x^3-2x^2-x+2|\,dx &= \int_0^1 x^3-2x^2-x+2\,dx + \int_1^2 -(x^3-2x^2-x+2)dx + \int_2^4 x^3-2x^2-x+2\,dx \\
&= \left[\frac{x^4}{4}-\frac{2x^3}{3}-\frac{x^2}{2}+2x\right]_0^1 + \left[\frac{x^4}{4}-\frac{2x^3}{3}-\frac{x^2}{2}+2x\right]_2^1 + \left[\frac{x^4}{4}-\frac{2x^3}{3}-\frac{x^2}{2}+2x\right]_2^4 \\
&= 2\times\left(\frac{1}{4}-\frac{2}{3}-\frac{1}{2}+2\right) - 2\times\left(\frac{16}{4}-\frac{16}{3}-\frac{4}{2}+4\right) + \left(\frac{256}{4}-\frac{128}{3}-\frac{16}{2}+8\right) \\
&= 2\times\frac{3-8-6+24}{12} - 2\times\left(4-\frac{16}{3}-2+4\right) + \left(64-\frac{128}{3}-8+8\right) \\
&= \frac{13}{6}-\frac{4}{3}+64-\frac{128}{3} \\
&= -\frac{251}{6}+64 \\
&= \frac{133}{6}
\end{aligned}
$$

(ii) Consider the following graph:



Between 0 and $3\pi/4$, both $\sin x$ and $\cos x$ are positive, therefore the integrand is $\sin x + \cos x$. From $3\pi/4$ to π, $\cos x$ is negative and its magnitude is greater than $\sin x$, so the integrand is $-(\sin x + \cos x)$. Between $-\pi/4$ and 0, $\cos x$ is positive and its magnitude is greater than $\sin x$, so the integrand is $\sin x + \cos x$. Between $-\pi/2$ and $-\pi/4$, $\sin x$ is negative and its magnitude is greater than $\cos x$, therefore the integrand is $-(\sin x + \cos x)$. Lastly, between $-\pi$ and $-\pi/2$, both $\sin x$ and $\cos x$ are negative and therefore the integrand is again $-(\sin x + \cos x)$.

$$
\therefore |\sin x + \cos x| = \begin{cases} -(\sin x + \cos x) = -\pi \leqslant x \leqslant \dfrac{-\pi}{4} \\[2ex] \sin x + \cos x = \dfrac{-\pi}{4} \leqslant x \leqslant \dfrac{3\pi}{4} \\[2ex] -(\sin x + \cos x) \;=\; \dfrac{3\pi}{4} \leqslant x \leqslant \pi \end{cases}
$$

$$
\begin{aligned}
\therefore \int_{-\pi}^{\pi} |\sin x + \cos x|\ dx &= -\int_{-\pi}^{-\pi/4} \sin x + \cos x\ dx + \int_{-\pi/4}^{3\pi/4} \sin x + \cos x\ dx - \int_{3\pi/4}^{\pi} \sin x + \cos x\ dx \\
&= -\left[-\cos x + \sin x\right]_{-\pi}^{-\pi/4} + \left[-\cos x + \sin x\right]_{-\pi/4}^{3\pi/4} - \left[-\cos x + \sin x\right]_{3\pi/4}^{\pi} \\
&= -2 \times \left(-\frac{1}{\sqrt{2}} + \frac{-1}{\sqrt{2}}\right) - -(- -1 + 0) + 2 \times \left(-\frac{-1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - (- -1 + 0) \\
&= \frac{4}{\sqrt{2}} + 1 + \frac{4}{\sqrt{2}} - 1 \\
&= 4\sqrt{2}
\end{aligned}
$$

Comments

Another fair albeit a long question. I can offer little wisdom. You can see that, as usual, the cubic was easy to factorise. I got that integral right, miraculously, when I came to type it up. Splitting up $|\sin x + \cos x|$ into branches was a chore, but likely I made a meal of it. Note that I held my nerve and multiplied some of the expressions by two to save myself a little labour.

Question 37

$$\frac{1}{2} = \frac{1}{3} + \frac{1}{6}$$

$$\frac{1}{3} = \frac{1}{4} + \frac{1}{12}$$

$$\frac{1}{4} = \frac{1}{5} + \frac{1}{20}$$

$$\frac{1}{5} = \frac{1}{6} + \frac{1}{30}$$

$$\vdots$$

$$\frac{1}{N} = \frac{1}{N+1} + \frac{1}{N(N+1)}$$

All that is needed for a proof is to check the following:

$$\frac{1}{N+1} + \frac{1}{N(N+1)} = \frac{N}{N(N+1)} + \frac{1}{N(N+1)} = \frac{N+1}{N(N+1)} = \frac{1}{N}$$

$$\begin{aligned}
\frac{1}{N} &= \frac{1}{a} + \frac{1}{b} \\
ab &= bN + aN \\
&= (a+b)N \\
ab - (a+b)N &= 0 \\
ab - (a+b)N + N^2 &= N^2 \\
(a-N)(b-N) &= N^2
\end{aligned}$$

Therefore we have, provided a, b and N are all non-zero, the following statement:

$$\frac{1}{N} = \frac{1}{a} + \frac{1}{b} \iff (a-N)(b-N) = N^2$$

Suppose that N^2 is prime, then N^2 can only be factorised in two ways:

$$1 \times N^2 \quad N \times N$$

From the previous statement we know that the fraction $1/N$ can be expressed as the sum of the two fractions $1/a$ and $1/b$ if and only if $(a-N)(b-N) = N^2$. Therefore we have two cases. Either $(a-N) = N$ and $(b-N) = N$, in which case $a = b = 2N$, or, without loss of generality $(a-N) = 1$ and $(b-N) = N^2$, in which case $a = N+1$ and $b = N(N+1)$. These are the only two cases and if we preclude the first by stipulating that a and b must be distinct, only the one case remains, namely:

$$\frac{1}{N} = \frac{1}{N+1} + \frac{1}{N(N+1)}$$

If we multiply through we get:

$$\frac{2}{N} = \frac{2}{N+1} + \frac{2}{N(N+1)}$$

We know that for any prime N this identity is unique. Furthermore, if N is prime and $N > 2$ then N is odd, hence N is even and we can write the above unique identity thus:

$$\frac{2}{N} = \frac{1}{(N+1)/2} + \frac{1}{N(N+1)/2}$$

Comments

The last time I answered this question with a student we took a while to get the reasoning right for the $1/N$ case, but we got there in the end. It is a subtle argument and I was left wondering who thought it up in the first place.

Question 38

$$\begin{aligned}
p(x) &= (x-a)^2 q(x) \\
p'(x) &= 2(x-a)q(x) + (x-a)^2 q'(x) \\
&= (x-a)\left[2q(x) + (x-a)q'(x)\right] \\
\therefore p'(a) &= (a-a)\left[\cdots\right] = 0
\end{aligned}$$

$$\begin{aligned}
p(x) &= (x-a)^4 q(x) \\
p'(x) &= 4(x-a)^3 q(x) + (x-a)^4 q'(x) \\
&= (x-a)^3\left[4q(x) + (x-a)q'(x)\right] \\
\therefore p'(a) &= (a-a)^3\left[\cdots\right] = 0
\end{aligned}$$

We write the polynomial in the form $p(x) = (x-a)^4 q_0(x)$. Then:

$$\begin{aligned}
p'(x) &= (x-a)^3 q_1(x) \\
p''(x) &= (x-a)^2 q_2(x) \\
p'''(x) &= (x-a) q_3(x)
\end{aligned}$$

Now:

$$\begin{aligned}
p'''(x) &= 6(5)(4)x^3 + 4(5)(4)(3)x^2 - 5(4)(3)(2)x - 40(3)(2)(1) \\
&= 2(3)(5)x^3 + 4(5)(3)x^2 - 5(3)(2)x - 10(3)(2)(1) \\
&= 2(5)x^3 + 2(2)(5)x^2 - 5(2)x - 10(2)(1) \\
&= 5x^3 + 2(5)x^2 - 5x - 2(5)(1) \\
&= x^3 + 2x^2 - x - 2
\end{aligned}$$

Setting $x = a$ and making use of the fact that $p'''(a) = 0$ we have:

$$a^3 + 2a^2 - a - 2 = 0$$

By inspection $a = 1$ is a root. Dividing by $a - 1$ gives $a^2 + 3a + 2$ and factorising by inspection gives $a = -1$ and $a = -2$ as the other two roots. Since k and m remain unknown, we consider $p''(x)$:

$$\begin{aligned}
p''(x) &= 6(5)x^4 + 4(5)(4)x^3 - 5(4)(3)x^2 - 40(3)(2)x - 40(2)(1) \\
&= 30x^4 + 80x^3 - 60x^2 - 240x - 80 \\
&= 10(3x^4 + 8x^3 - 6x^2 - 24x - 8)
\end{aligned}$$

We then see which value of a results in $p''(a) = 0$:

$$p''(1) = 10(3 + 8 - 6 - 24 - 8) \neq 0$$

$$p''(-1) = 10(3 - 8 - 6 + 24 - 8) \neq 0$$

$$p''(-2) = 10(3(16) + 8(-8) - 6(4) - 24(-2) - 8) = 10(48 - 64 - 24 + 48 - 8) = 0$$

Thus $a = -2$. Now we find k by considering $p'(x)$:

$$\begin{aligned} p'(x) &= 6x^5 + 4(5)x^5 - 5(4)x^3 - 40(3)x^2 - 40(2)x + k \\ &= 6x^5 + 20x^4 - 20x^3 - 120x^2 - 80x + k \\ p'(-2) &= 6(-32) + 20(16) - 20(-8) - 120(4) - 80(-2) + k \\ &= -192 + 320 + 160 - 480 + 160 + k \\ &= -32 + k \end{aligned}$$

Therefore since $p(-2) = 0$, $k = 32$.

Finally we use $p(-2) = 0$ to get m:

$$\begin{aligned} p(-2) = 0 &= 64 + 4(-32) - 5(16) - 40(-8) - 40(4) + 32(-2) + m \\ &= 64 - 128 - 80 + 320 - 160 - 64 + m \\ &= -48 + m \end{aligned}$$

Thus $m = 48$.

Comments

Again I have little to say about this question aside from the fact that due care with the arithmetic makes the working out a lot less onerous and therefore a good deal less error prone.

Question 39

$$\begin{aligned}\sin\theta &= 2\sin{}^{\theta}\!/_{2}\cos{}^{\theta}\!/_{2}\\ &= \cos^2{}^{\theta}\!/_{2}(2\tan{}^{\theta}\!/_{2})\\ &= \frac{2\tan{}^{\theta}\!/_{2}}{\sec^2{}^{\theta}\!/_{2}}\\ &= \frac{2\tan{}^{\theta}\!/_{2}}{1+\tan^2{}^{\theta}\!/_{2}}\end{aligned}$$

$$\begin{aligned}\cos\theta &= \cos^2{}^{\theta}\!/_{2} - \sin^2{}^{\theta}\!/_{2}\\ &= \cos^2{}^{\theta}\!/_{2}(1-\tan^2{}^{\theta}\!/_{2})\\ &= \frac{1-\tan^2{}^{\theta}\!/_{2}}{\sec^2{}^{\theta}\!/_{2}}\\ &= \frac{1-\tan^2{}^{\theta}\!/_{2}}{1+\tan^2{}^{\theta}\!/_{2}}\end{aligned}$$

$$\begin{aligned}\frac{1+\cos\theta}{\sin\theta} &= \frac{1+\left(\dfrac{1-t^2}{1+t^2}\right)}{\dfrac{2t}{1+t^2}}\\ &= \frac{1+t^2+1-t^2}{2t}\\ &= \frac{1}{t}\\ &= \frac{\cos{}^{\theta}\!/_{2}}{\sin{}^{\theta}\!/_{2}}\\ &= \frac{\sin\left({}^{\pi}\!/_{2}-{}^{\theta}\!/_{2}\right)}{\cos\left({}^{\pi}\!/_{2}-{}^{\theta}\!/_{2}\right)}\\ &= \tan\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\end{aligned}$$

$$\begin{aligned}t &= \tan\frac{\theta}{2}\\ dt &= \frac{1}{2}\sec^2\frac{\theta}{2}\,d\theta = \frac{1}{2}(1+t^2)\,d\theta\end{aligned}$$

$$\begin{aligned}\therefore I &= \int_0^1 \frac{1}{1+\dfrac{\cos\alpha\,2t}{1+t^2}}\;2(1+t^2)^{-1}dt\\ &= \int_0^1 \frac{1+t^2}{1+t^2+2t\cos\alpha}\frac{2}{1+t^2}\,dt\\ &= \int_0^1 \frac{2}{1+t^2+2t\cos\alpha}\,dt\\ &= \int_0^1 \frac{2}{\cos^2\alpha+\sin^2\alpha+t^2+2t\cos\alpha}\,dt\\ &= \int_0^1 \frac{2}{(t+\cos\alpha)^2+\sin^2\alpha}\,dt\end{aligned}$$

$$t + \cos\alpha = \sin\alpha \tan u$$

$$dt = \sin\alpha \sec^2 u \; du$$

We leave the limits of integration for the time being:

$$\begin{aligned}
\therefore I &= \int \frac{2}{\sin^2\alpha \tan^2 u + \sin^2\alpha} \; \sin\alpha \sec^2 u \; du \\
&= \int \frac{2}{\sin^2\alpha(1+\tan^2 u)} \; \sin\alpha(1+\tan^2)u \; du \\
&= \int \frac{2\sin\alpha}{\sin^2\alpha} \; du \\
&= \frac{2}{\sin\alpha}[u]
\end{aligned}$$

Returning to the limits of integration:

When $t = 0$:

$$\begin{aligned}
\cos\alpha &= \sin\alpha \tan u \\
\tan u &= \cot\alpha \\
&= \tan\left(\frac{\pi}{2} - \alpha\right) \\
\therefore u &= \frac{\pi}{2} - \alpha
\end{aligned}$$

When $t = 1$:

$$\begin{aligned}
1 + \cos\alpha &= \sin\alpha \tan u \\
\tan u &= \frac{1+\cos\alpha}{\sin\alpha} \\
&= \tan\left(\frac{\pi}{2} - \frac{\alpha}{2}\right) \\
\therefore u &= \frac{\pi}{2} - \frac{\alpha}{2}
\end{aligned}$$

$$\therefore I = \frac{2}{\sin\alpha}\left[\left(\frac{\pi}{2} - \frac{\alpha}{2}\right) - \left(\frac{\pi}{2} - \alpha\right)\right] = \frac{\alpha}{\sin\alpha}$$

For the last integral we replace α with β because we will need a substitution:

$$\tilde{I} = \int_0^{\pi/2} \frac{1}{1+\sin\beta\cos\phi} \; d\phi$$

The limits of integration suggest that the following identity should be used in a substitution for ϕ:

$$\sin\theta = -\cos\left(\theta - \frac{\pi}{2}\right)$$

We therefore set $\phi = \theta - \dfrac{\pi}{2}$ so that $\cos\phi = -\sin\theta$ and $d\phi = d\theta$:

$$\tilde{I} = \int_{-\pi/2}^{0} \frac{1}{1 + \sin\beta(-\sin\theta)}\ d\theta$$

We can fix the limits of integration by a further substitution $-\theta$ for θ:

$$\tilde{I} = \int_{-\pi/2}^{0} \frac{1}{1 + \sin\beta\sin\theta}\ - d\theta = \int_{0}^{\pi/2} \frac{1}{1 + \sin\beta\sin\theta}\ d\theta$$

We now need a substitution for $\sin\beta$ that avoids any further change of sign. We note:

$$\sin\beta = \cos\left(\beta - \frac{\pi}{2}\right)$$

Therefore we set $\alpha = \beta - \dfrac{\pi}{2}$ so that $\sin\beta = \cos\alpha$. Then we have:

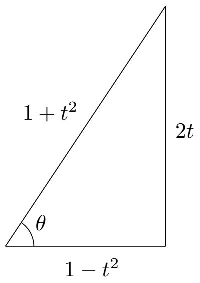
$$\begin{aligned}\tilde{I} &= I \\ &= \frac{\beta}{\sin\beta} \\ &= \frac{\alpha - \frac{\pi}{2}}{\sin\left(\alpha - \frac{\pi}{2}\right)} \\ &= \frac{\frac{\pi}{2} - \alpha}{\cos\alpha}\end{aligned}$$

Comments

There is a geometric argument that suffices to prove the first two identities which makes use of the usual identity:

$$\tan\theta = \frac{2t}{1-t^2}$$

Here, $t = \tan {}^{\pi}\!/_{2}$, of course. If we consider the following right angled triangle, then the identities for $\sin\theta$ and $\cos\theta$ are immediate:



The only trouble with this argument and the reason I did not give it in the answer is that it only works for $0 \leqslant \theta < {}^{\pi}\!/_{2}$. It can be made to work for the other quadrants, but it is not worth the effort.

In the last part you need to be very careful. The change of variable from θ to ϕ suggests a substitution and the limits should have given you a clue as to what it should have been. I changed the α to a β because I was getting confused. Possibly the reason why the examiner did not do so was because changing this variable does not affect the limits of integration.

What else is there to say about this question? Only that the examiners have been very kind. In fact, this is actually mentioned in the accompanying comments. If you found the question difficult, perhaps this observation is no help to you, but nonetheless I hope it gives you hope.

Question 40

$$\mathbf{r}_1 = \lambda \begin{pmatrix} \cos\theta + \sqrt{3} \\ \sqrt{2}\sin\theta \\ \cos\theta - \sqrt{3} \end{pmatrix} \qquad \mathbf{r}_2 = \mu \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\begin{aligned}
\cos\theta &= \frac{\mathbf{r}_1.\mathbf{r}_2}{r_1 r_2} \\
&= \frac{\lambda\mu\left(a\cos\theta + a\sqrt{3} + b\sqrt{2}\sin\theta + c\cos\theta - c\sqrt{3}\right)}{\lambda\mu\left[(\cos\theta+\sqrt{3})^2 + 2\sin^2\theta + (\cos\theta-\sqrt{3})^2\right]^{1/2}\left(a^2+b^2+c^2\right)^{1/2}} \\
&= \frac{a\cos\theta + a\sqrt{3} + b\sqrt{2}\sin\theta + c\cos\theta - c\sqrt{3}}{\left(\cos^2\theta + 2\cos\theta\sqrt{3} + 3 + 2\sin^2\theta + \cos^2\theta - 2\cos\theta\sqrt{3} + 3\right)^{1/2}\left(a^2+b^2+c^2\right)^{1/2}} \\
&= \frac{a\cos\theta + a\sqrt{3} + b\sqrt{2}\sin\theta + c\cos\theta - c\sqrt{3}}{\sqrt{2}\left(\cos^2\theta + 3 + \sin^2\theta\right)^{1/2}\left(a^2+b^2+c^2\right)^{1/2}} \\
&= \frac{a\cos\theta + a\sqrt{3} + b\sqrt{2}\sin\theta + c\cos\theta - c\sqrt{3}}{2\sqrt{2}\left(a^2+b^2+c^2\right)^{1/2}} \\
\therefore \theta &= \cos^{-1}\left(\frac{a\cos\theta + a\sqrt{3} + b\sqrt{2}\sin\theta + c\cos\theta - c\sqrt{3}}{2\sqrt{2}\left(a^2+b^2+c^2\right)^{1/2}}\right)
\end{aligned}$$

The second line $\mathbf{r}_2$ is an arbitrary line through the origin and $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$, so:

$$2\left(a\cos\theta + a\sqrt{3} + b\sqrt{2}\sin\theta + c\cos\theta - c\sqrt{3}\right) = 2\sqrt{2}\sqrt{3}\left(a^2+b^2+c^2\right)^{1/2}$$

$$a\cos\theta + a\sqrt{3} + b\sqrt{2}\sin\theta + c\cos\theta - c\sqrt{3} = \sqrt{6}\left(a^2+b^2+c^2\right)^{1/2}$$

In order for this equation to hold regardless of the value of θ, we require the following expression involving θ to be constant:

$$a\cos\theta + b\sqrt{2}\sin\theta + c\cos\theta$$

Since this can be written in the form $R\sin(\theta+\theta_0)$, in order for it to be constant it must be equal to zero. Hence we must have $a = -c$ and $b = 0$. Since $\mathbf{r}_2$ scales with μ we are free to choose $a = 1$. Therefore the line m is:

$$\mathbf{r}_2 = \mu \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Since the angle between the lines l and m is fixed, the line l will trace out a cone with axis m as θ varies. Therefore in order for the intersection of the cone to be a circle, we require the axis of the cone to be perpendicular to the given plane. Now the equation of a plane is given by the following vector equation:

$$(\mathbf{r} - \mathbf{a}).\hat{\mathbf{n}} = 0$$

Here $\mathbf{r}$ is an variable point on the plane, $\mathbf{a}$ a fixed point and $\hat{\mathbf{n}}$ the normal. Dropping the requirement that the normal be a unit vector, in other words $\mathbf{n} = \lambda\hat{\mathbf{n}}$, we have:

$$(\mathbf{r} - \mathbf{a}).\mathbf{n} = 0$$

$$\mathbf{r}.\mathbf{n} = \mathbf{a}.\mathbf{n}$$

Writing the given equation in vector form:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} . \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 4\sqrt{3}$$

So we see immediately that the axis of the cone is indeed normal to the plane and therefore their intersection forms a circle.

Now we find the centre of the circle. We know that the line m passes through the plane, therefore we make use of the above equation:

$$\mu \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} . \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 4\sqrt{3}$$

Thus $2\mu = 4\sqrt{3}$ and so $\mu = 2\sqrt{3}$ and the centre C is:

$$\begin{pmatrix} 2\sqrt{3} \\ 0 \\ -2\sqrt{3} \end{pmatrix}$$

Now we find a point on the circle. We are free to choose any value for θ so we choose $\pi/2$ so that $\cos\theta = 0$. Then the equation of the line l becomes:

$$\lambda \begin{pmatrix} \sqrt{3} \\ \sqrt{2} \\ -\sqrt{3} \end{pmatrix}$$

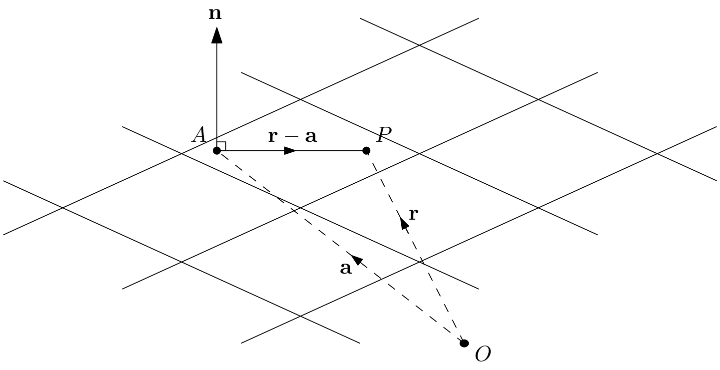
By inspection we can see that $\lambda = 2$ will give us a point A on the plane. Finally:

$$\begin{aligned} r &= |A - C| \\ &= \left| \begin{pmatrix} 2\sqrt{3} \\ 0 \\ -2\sqrt{3} \end{pmatrix} - \begin{pmatrix} 2\sqrt{3} \\ 2\sqrt{2} \\ -2\sqrt{3} \end{pmatrix} \right| \\ &= \left| \begin{pmatrix} 0 \\ -2\sqrt{2} \\ 0 \end{pmatrix} \right| \\ &= 2\sqrt{2} \end{aligned}$$

Comments

This question is all about understanding basic vector geometry and, for me at least, being comfortable with the notions of lines and planes in three dimensions. In particular, you need to know both the vector equations that define lines and planes together with their geometric interpretations. A very good understanding of the vector equation of the plane is especially important in its own right and because it leads to everything else.

I hope I made a good stab at communicating my understanding of a plane in my answer but it is worth going through it again now. So, in order to define a plane you need to supply the normal to it together with one point that it contains. Then you can reason that the position of any point within it relative to the given point must be orthogonal to the normal:



Thus we get the following vector equation:

$$(\mathbf{r} - \mathbf{a}).\mathbf{n} = 0$$

This is the important one that you must have a good handle on. It leads immediately to the more commonplace one:

$$\mathbf{r}.\mathbf{n} = \mathbf{a}.\mathbf{n}$$

However, I find that some students have learned this second one without knowing how it is derived from the first, and therefore they lack the necessary geometric intuition. I cannot help but think that this question is all about uncovering whether the student has such an intuition, however.

Moving on, I find equations of the form...

$$x - z = 4\sqrt{3}$$

...to be a good deal less helpful than equations of the form:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} . \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 4\sqrt{3}$$

The reason for this is that whilst the right hand side carries with it no particular meaning, geometric or otherwise, the left hand side certainly does. In my answer I do the conversion.

Finally, note that I go to some lengths to try to make it clear to the examiner that I understand what is going on. I feel this buys me some leniency. I do not derive the actual equation of the circle, for example, I just reason confidently that it must be so. I think I got it right, because you are asked to find the radius in the last bit. If you had been asked to derive the equation of the circle explicitly, asking you for the radius afterwards would not really have made sense. I was glad that I was not required to derive the equation explicitly, actually, it might have gotten messy.

Question 41

$$
\begin{aligned}
(x^2 - ax + b)(x^2 + ax + c) &= x^4 + ax^3 + cx^2 - ax^3 - a^2x^2 - acx + bx^2 + abx + bc \\
&= x^4 + (-a^2 + b + c)x^2 + (-ac + ab)x + bc
\end{aligned}
$$

Equating coefficients:

$$
\left.\begin{aligned}
p &= -a^2 + b + c \\
q &= ab - ac \\
r &= bc
\end{aligned}\right\}
$$

$$
\begin{aligned}
u^3 + 2pu^2 + (p^2 - 4r)u - q^2 &= u^3 + 2(-a^2 + b + c)u^2 + \left[(-a^2 + b + c)^2 - 4bc\right]u - (ab - ac)^2 \\
&= u^3 + 2(-a^2 + b + c)u^2 \\
&\quad + (a^4 + b^2 + c^2 - 2a^2b - 2a^2c + 2bc - 4bc)u \\
&\quad - a^2b^2 + 2a^2bc - a^2c^2
\end{aligned}
$$

Substituting a^2 for u:

$$
a^6 - 2a^6 + 2a^4b + 2a^4c + a^6 + a^2b^2 + a^2c^2 - 2a^4b - 2a^2c - 2a^2bc - a^2b^2 + 2a^2bc - a^2c^2 = 0
$$

Setting $p = -1$, $q = -6$ and $r = 15$ and then substituting 9 for u:

$$
\begin{aligned}
u^3 + 2(-1)u^2 + \left[(-1)^2 - 4(15)\right]u - (-6)^2 &= u^3 + 2u^2 - 59u - 36 \\
&= 729 - 2 \times 81 - 59 \times 9 - 36 \\
&= 729 - 162 - 531 - 36 \\
&= 0
\end{aligned}
$$

We try substituting $x + t$ for y and choose t in order to eliminate the cubic term:

$$
\begin{aligned}
(x + t)^4 - 8(x + t)^3 + \cdots &= 0 \\
x^4 + 4tx^3 + \cdots - 8x^3 - \cdots &= 0
\end{aligned}
$$

So we require:

$$
\begin{aligned}
4tx^3 - 8x^3 &= 0 \\
4t - 8 &= 0
\end{aligned}
$$

Which gives $t = 2$. Substituting fully:

$$
\begin{aligned}
(x + 2)^4 - 8(x + 2)^3 + 23(x + 2)^2 - 34(x + 2) + 39 &= x^4 + 4x^3(2) + 6x^2(2)^2 + 4x(2)^3 + (2)^4 \\
&\quad - 8\left[x^3 + 3x^2(2) + 3x(2)^2 + (2)^3\right] \\
&\quad + 23\left[x^2 + 4x + 4\right] \\
&\quad - 34\left[x + 2\right] \\
&\quad + 39
\end{aligned}
$$

$$
\begin{aligned}
&= x^4 + 8x^3 + 24x^2 + 32x + 16 \\
&\quad - 8x^3 - 48x^2 - 96x - 64 \\
&\quad + 23x^2 + 92x + 92 \\
&\quad - 34x - 68 \\
&\quad + 39 \\
&= x^4 - x^2 - 6x + 15
\end{aligned}
$$

This is the original equation with $p = -1$, $q = -6$ and $r = 15$. Therefore we have the following simultaneous equations for a, b and c:

$$
\left.
\begin{aligned}
-1 &= -a^2 + b + c \\
-6 &= ab - ac \\
15 &= bc
\end{aligned}
\right\}
$$

We can solve these directly, however, we know that a^2 is a root of the previous cubic equation involving u and that with p, q and r as given that $u = 9$ is a root of that equation. Therefore we are free to choose $a = 3$ or $a = -3$. The symmetry on the right hand side of the original equation suggests the choice is arbitrary. Therefore:

$$
\left.
\begin{aligned}
-1 &= -9 + b + c \\
-6 &= -3b - -3c \\
15 &= bc
\end{aligned}
\right\}
$$

Simplifying:

$$
\left.
\begin{aligned}
8 &= b + c \\
2 &= b - c \\
15 &= bc
\end{aligned}
\right\}
$$

By inspection we have $b = 5$, $c = 3$. Therefore we only have to factorise first $x^2 + 3x + 5$...

$$
\begin{aligned}
0 &= x^2 + 3x + 5 \\
&= x^2 + 3x + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 + 5 \\
&= \left(x + \frac{3}{2}\right)^2 - \frac{9}{4} + 5 \\
&= \left(x + \frac{3}{2}\right)^2 - \frac{9}{4} + \frac{20}{4} \\
&= \left(x + \frac{3}{2}\right)^2 + \frac{11}{4}
\end{aligned}
$$

$$
\left(x + \frac{3}{2}\right)^2 = \frac{-11}{4}
$$

$$\therefore x = \frac{-3 \pm i\sqrt{11}}{2}$$

...and then $x^2 - 3x + 3$:

$$\begin{aligned}
0 &= x^2 - 3x + 3 \\
&= x^2 - 3x + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 + 3 \\
&= \left(x - \frac{3}{2}\right)^2 - \frac{9}{4} + 3 \\
&= \left(x - \frac{3}{2}\right)^2 - \frac{9}{4} + \frac{12}{4} \\
&= \left(x - \frac{3}{2}\right)^2 + \frac{3}{4} \\
\left(x - \frac{3}{2}\right)^2 &= \frac{-3}{4}
\end{aligned}$$

Finally, we recall that $y = x + 2$ and so:

$$y = \frac{1 \pm i\sqrt{11}}{2}, \quad \frac{7 \pm i\sqrt{3}}{2}$$

Comments

I assume that you have been reading the comments that accompany the questions as you go along but I especially commend this one to you. The reason, apart from the fact that it is very amusing, is that it makes clear that there is a specific method to this question and a pretty ingenious one at that. I think it makes the point nicely that you are often led through the answer and are required to have little insight yourself. Under examination conditions, this is a good thing.

Even outside of an examination I initially failed to grasp the relevance of the cubic involving u. Indeed I asked myself, why can we not simply solve the simultaneous equations to find a, b and c? So I tried this, eliminating c and then a, but ended up with a quartic for b. By symmetry, eliminating c and then a would have amounted to the same thing. Trying to leave myself with an equation involving a, I got this far:

$$\left.\begin{aligned} -ab &= -a^3b + b^2 + 15a \\ -6b &= ab^2 - 15a \end{aligned}\right\}$$

I could not manage to separate a from b from here, though. So the ingenuity in the method, at least for me, lies not in the removal of the cubic term from a quartic equation, something you have to work out for yourself, by the way, nor in the factorisation of a quartic without a cubic term, but in the derivation of the aforementioned corresponding cubic equation which allows you to find a^2 and hence simplify the simultaneous equations in order to find b and c.

Question 42

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)}$$

$$= \frac{\sin A\cos B + \cos A\sin B}{\cos A\cos B - \sin A\sin B}$$

$$= \frac{\tan A\cos B + \sin B}{\cos B - \tan A\sin B}$$

$$= \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

Substituting θ for A and 2θ for B we get:

$$\tan 3\theta = \frac{\tan\theta + \tan 2\theta}{1 - \tan\theta\tan 2\theta}$$

Substituting $\dfrac{2\tan\theta}{1-\tan^2\theta}$ for $\tan 2\theta$:

$$\tan 3\theta = \frac{\tan\theta + \dfrac{2\tan\theta}{1-\tan^2\theta}}{1 - \dfrac{\tan\theta.2\tan\theta}{1-\tan^2\theta}}$$

$$= \frac{\tan\theta(1-\tan^2\theta) + 2\tan\theta}{1 - \tan^2\theta - 2\tan^2\theta}$$

$$= \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

Since $0 \leqslant \theta < \pi/2$ we know that $\sin\theta$ is positive. Therefore:

$$\sin\theta = \sqrt{1-\cos^2\theta}$$

$$= \sqrt{1 - \frac{11}{5}}$$

$$= \frac{1}{\sqrt{5}}$$

$$\tan\theta = \frac{1}{\sqrt{5}} / \frac{2}{\sqrt{5}} = \frac{1}{2}$$

$$\tan 3\theta = \frac{3/2 - 1/8}{1 - 3/4} = \frac{11/8}{1/4} = \frac{11}{2}$$

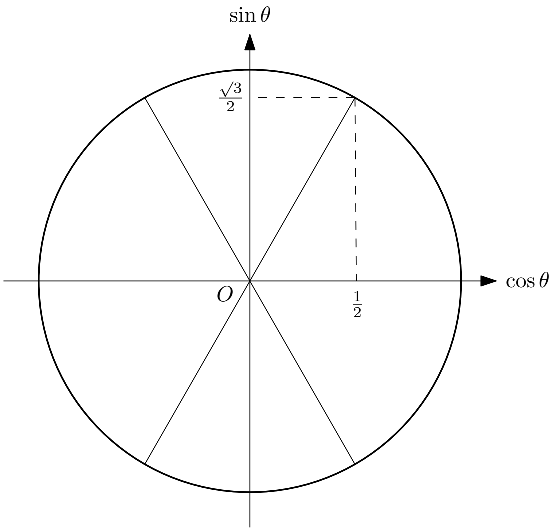
(i) We note that if $\tan A = \tan B$ then $A = B + n\pi$, $n \in \mathbb{Z}$.

$$\therefore 3\cos^{-1} x = 3\cos^{-1}\left(\frac{2}{\sqrt{5}}\right) + n\pi$$

$$= \cos^{-1}\left(\frac{2}{\sqrt{5}}\right) + \frac{n\pi}{3}$$

$$
\begin{aligned}
\therefore x &= \cos\left(\cos^{-1}\left(\frac{2}{\sqrt{5}}\right) + \frac{n\pi}{3}\right) \\
&= \cos\left(\theta + \frac{n\pi}{3}\right) \\
&= \cos\theta\cos\left(\frac{n\pi}{3}\right) - \sin\theta\sin\left(\frac{n\pi}{3}\right) \\
&= \frac{2}{\sqrt{5}}\cos\left(\frac{n\pi}{3}\right) - \frac{1}{\sqrt{5}}\sin\left(\frac{n\pi}{3}\right) \\
&= \frac{2}{\sqrt{5}}\left[\cos\left(\frac{n\pi}{3}\right) - \sin\left(\frac{n\pi}{3}\right)\right]
\end{aligned}
$$

There are six distinct solutions as $\dfrac{n\pi}{3}$ moves around the unit circle:



$$
\begin{aligned}
x =& \frac{1}{\sqrt{5}}(2-0) = \frac{2}{\sqrt{5}}, \\
& \frac{1}{\sqrt{5}}\left(1 - \frac{\sqrt{3}}{2}\right) = \frac{2-\sqrt{3}}{2\sqrt{5}}, \\
& \frac{1}{\sqrt{5}}\left(-1 - \frac{\sqrt{3}}{2}\right) = \frac{-2-\sqrt{3}}{2\sqrt{5}}, \\
& \frac{1}{\sqrt{5}}(-2-0) = \frac{-2}{\sqrt{5}}, \\
& \frac{1}{\sqrt{5}}\left(1 + \frac{\sqrt{3}}{2}\right) = \frac{2+\sqrt{3}}{2\sqrt{5}}, \\
& \frac{1}{\sqrt{5}}\left(-1 + \frac{\sqrt{3}}{2}\right) = \frac{-2+\sqrt{3}}{2\sqrt{5}}
\end{aligned}
$$

(ii)

$$\cos\left(\frac{1}{3}\tan^{-1}y\right) = \frac{2}{\sqrt{5}}$$

$$\cos\left(\frac{1}{3}\tan^{-1}y\right) = \cos\theta$$

$$\frac{1}{3}\tan^{-1}y = 2n\pi \pm \theta$$

$$\tan^{-1}y = 6n\pi \pm 3\theta$$

$$y = \tan\left(6n\pi \pm 3\theta\right)$$

Since tan is periodic with period π, we can ignore the $6n\pi$. Finally:

$$\begin{aligned} y &= \tan(\pm 3\theta) \\ &= \pm\tan(3\theta) \\ &= \pm\frac{11}{2} \end{aligned}$$

Comments

I thought this was a pretty difficult question, although the first part is just bookwork. There is no need to start as far back as I did, starting from the identity for $\tan(A+B)$ would have been fine. Note that I explicitly state that $\sin\theta$ is positive for the given interval, just to be on the safe side.

It was the second part that caused both me and my student some difficulty. It all looks so easy when it is typed up but believe me, we made hard work of it. We got the important bit, namely that if you state that $\tan A = \tan B$ then you have to state that $A = B + n\pi$ and not just $A = B$. After this, though, for some reason we could not quite get it right. Eventually we figured it out, but then evaluating the expression for x for all the different values of n again caused us some problems. I drew a table. When I came to type it up, I realised that considering the unit circle is probably by far the most elegant way to both show that there were six distinct values and figure them all out.

Mercifully the last part fell out fine. The only bit that requires a little thought, assuming of course that you realise what you need to do, is the bit where you can discard the $6n\pi$. If you got anywhere near a complete answer in a reasonable amount of time, I would say that you did extremely well.

Question 43

Stationary points are when $\dfrac{dy}{dx} = 0$ and since the exponential term cannot be zero we have:

$$x(1 - x^2) = 0$$

Hence $x = 0$ or $x = \pm 1$. Differentiating again:

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \left((x - x^3)e^{-x^2}\right)' \\
&= (1 - 3x^2)e^{-x^2} - 2x(x - x^3)e^{-x^2} \\
&= (1 - 5x^2 + 2x^4)e^{-x^2} \\
\left.\frac{d^2y}{dx^2}\right|_{x=0} &= (1)e^0 = 1 > 0
\end{aligned}$$

Hence the stationary point at $x = 0$ is a minimum. We are told that the curve passes through the origin and so the maximum is at the origin. Similarly:

$$\left.\frac{d^2y}{dx^2}\right|_{x=1} = (1 - 5 + 2)e^{-1} = -2e^{-1} < 0$$

So the stationary point at $x = 1$ is a maximum.

We note the following:

$$\begin{aligned}
\left(x^2e^{-x^2}\right)' &= 2xe^{-x^2} + x^2(-2x)e^{-x^2} \\
&= 2xe^{-x^2} + -2x^3e^{-x^2} \\
&= 2x(1 - x^2)e^{-x^2} \\
\int 2x(1 - x^2)e^{-x^2}\,dx &= x^2e^{-x^2} + c \\
\therefore y = \int x(1 - x^2)e^{-x^2}\,dx &= \frac{1}{2}x^2e^{-x^2} + c
\end{aligned}$$

Again y passes through the origin and therefore $c = 0$ and we have:

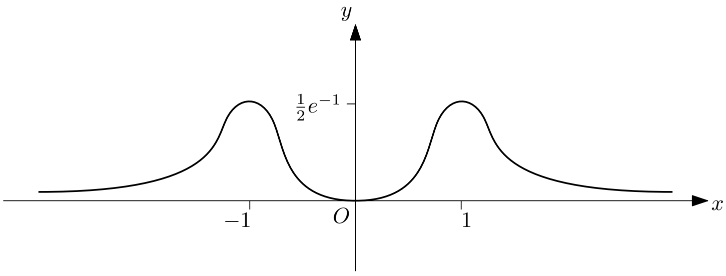
$$y = \frac{1}{2}x^2e^{-x^2}$$

Continuing with the maximum at $x = 1$, we note $y(1) = \frac{1}{2}(1)^2e^{-(1)^2} = \frac{1}{2}e^{-1}$, as required.

Also, since y depends only on x^2, we see straight away that $y(-1) = y(1)$ and therefore the coordinates of the last stationary point are:

$$\left(-1, \frac{1}{2}e^{-1}\right)$$

Similarly for $\dfrac{d^2y}{dx^2}$, therefore this stationary point is also a local maximum.



For the curve C_2, we see immediately that $y'(0) = 0$ and therefore, since we are given that $y(0) = 0$, there is a stationary point at the origin. Differentiating:

$$\begin{aligned}\frac{d^2y}{dx^2} &= \left((x - x^3)e^{-x^3}\right)' \\ &= (1 - 3x^2)e^{-x^3} - 3x^2(x - x^3)e^{-x^3} \\ &= (1 - 3x^2 - 3x^3 + 3x^5)e^{-x^3} \\ \left.\frac{d^2y}{dx^2}\right|_{x=0} &= (1)e^0 = 1 > 0\end{aligned}$$

Therefore the stationary point is a minimum.

Clearly also $y'(1) = 0$ and so there is a stationary point at $x = 1$. Furthermore:

$$\left.\frac{d^2y}{dx^2}\right|_{x=1} = (1 - 3 - 3 + 3)e^0 = -2 < 0$$

Therefore the stationary point is a maximum.

The condition on k amounts to saying that the curve C_2's maximum point at $x = 1$ is above that of C_1. Since they both go through the origin, in order to demonstrate that this is true we need only show that the gradient of C_1 is greater than the gradient of C_2 on the interval $(0, 1)$. Since $(1 - x^2)$ is positive on that interval and the other two factors are also positive, we are left to show that on $(0, 1)$ the following inequality holds:

$$e^{-x^3} > e^{-x^2}$$

Taking logs of both sides and making use of the fact that the log function is strictly increasing we have:

$$-x^3 > -x^2$$

That is:

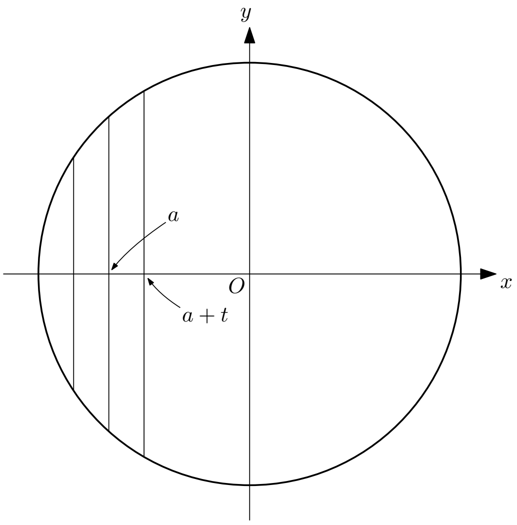
$$x^3 < x^2$$

And this is certainly true.

Comments

Hopefully a reasonably straightforward question. You are required to integrate y' and a good guess was some power of x multiplied by the exponential term up to a constant factor. There were some short cuts to be had, for example in realising the behaviour of both y and y'', specifically their dependency on only even powers of x. Was the hint that you were not required to find k strong enough to convince you not to try to integrate to find C_2, I wonder? To be honest, it was more the fact that I had enough to go on without integrating that persuaded me not to try.

Question 44



The area of a slice $A_{(-a,t)}$ of fixed width t at $x = -a$ is given by:

$$A_{(-a,t)} = 2\pi \int_{-a}^{-a+t} y \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2} dx$$

We set the radius to be r so that $x^2 + y^2 = r^2$ and taking y to be positive:

$$\begin{aligned} y &= \left(r^2 - x^2\right)^{1/2} \\ \frac{dy}{dx} &= \frac{1}{2}\left(r^2 - x^2\right)^{-1/2} . -2x \\ &= -x\left(r^2 - x^2\right)^{-1/2} \end{aligned}$$

Substituting back into the expression for $A_{(-a,t)}$:

$$\begin{aligned} A_{(-a,t)} &= 2\pi \int_{-a}^{-a+t} \left(r^2 - x^2\right)^{1/2} \left[1 + x^2(r^2 - x^2)^{-1}\right]^{1/2} dx \\ &= 2\pi \int_{-a}^{-a+t} \left(r^2 - x^2\right)^{1/2} \left[1 + \frac{x^2}{r^2 - x^2}\right]^{1/2} dx \\ &= 2\pi \int_{-a}^{-a+t} \left(r^2 - x^2\right)^{1/2} \left[\frac{r^2}{r^2 - x^2}\right]^{1/2} dx \\ &= 2\pi r \int_{-a}^{-a+t} dx \\ &= 2\pi r t \end{aligned}$$

The surface area of the slice is therefore independent of its position a and directly proportional to its width. The combined area of any number of slices of fixed with is therefore directly proportional to their number.

We calculate the volume $V_{(-a,t)}$ of a slice similarly:

$$\begin{aligned}
V_{(-a,t)} &= \pi \int_{-a}^{-a+t} y^2 \, dx \\
&= \pi \int_{-a}^{-a+t} (r^2 - x^2) \, dx \\
&= \pi \int_{-a}^{-a+t} r^2 \, dx - \pi \int_{a}^{a+t} x^2 \, dx \\
&= \pi r^2 t - \pi \left[\frac{x^3}{3} \right]_{-a}^{-a+t} \\
&= \pi r^2 t - \frac{\pi}{3} \left[(-a+t)^3 - (-a)^3 \right]_{-a}^{-a+t} \\
&= \pi r^2 t - \frac{\pi}{3} (-a^3 + 3a^2 t - 3at^2 + t^3 + a^3) \\
&= \pi r^2 t - \frac{\pi}{3} (3a^2 t - 3at^2 + t^3)
\end{aligned}$$

We want the ratio of the bread remaining, so we can set $a = r$:

$$\begin{aligned}
V_{(-r,t)} \equiv V_t &= \pi r^2 t - \frac{\pi}{3} (3r^2 t - 3rt^2 + t^3) \\
&= \pi r^2 t - \pi r^2 t + \pi r t^2 + \frac{\pi t^3}{3} \\
&= \pi r t^2 + \frac{\pi t^3}{3} \\
&= \frac{\pi t^2}{3} (3r - t)
\end{aligned}$$

Returning to the surface area:

$$A_{(-r,t)} \equiv A_t = 2\pi r t$$

We require $V_t < A_t$:

$$\begin{aligned}
\frac{\pi t^2}{3} (3r - t) &< 2\pi r t \\
t(3r - t) &< 6r \\
3rt - t^2 &< 6r \\
t^2 - 3rt + 6r &> 0 \\
t^2 - 3rt + \left(\frac{3r}{2} \right)^2 - \left(\frac{3r}{2} \right)^2 + 6r &> 0 \\
\left(t - \frac{3r}{2} \right)^2 - \left(\frac{3r}{2} \right)^2 + 6r &> 0
\end{aligned}$$

In order to guarantee that the left hand side is strictly positive, we have to have:

$$-\left(\frac{3r}{2}\right)^2 + 6r > 0$$

$$-\frac{9r^2}{4} + 6r > 0$$

$$-\frac{9r}{4} + 6 > 0$$

$$-\frac{3r}{4} + 2 > 0$$

$$-3r + 8 > 0$$

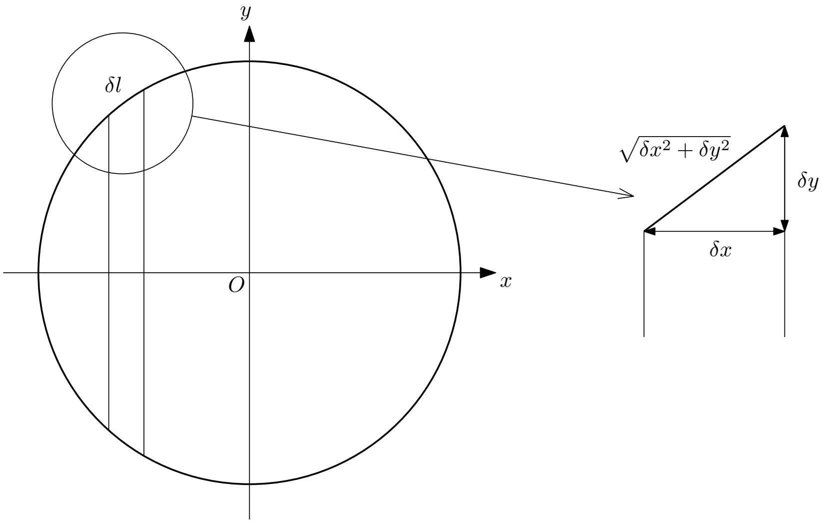
$$3r < 8$$

$$\therefore r < \frac{8}{3}$$

Comments

It is fair to say that we did not do this question by the quickest means in places, but we got it out. The fact that the surface area of a slice is not related to its position is perhaps counter-intuitive and therefore quite interesting.

In an earlier question I gave an ad-hoc proof for the volume of revolution since the question demanded it. Here the formulae for both the surface and volume of revolution are kindly given. Can you prove the formula for the surface of revolution, though? If you want to have a go, look away now.



The surface area of a slice of width δx is approximately the radius times the length δl. The surface area from $x = -a$ to $x = -a + t$ therefore is the sum of the surface areas of these slices:

$$\begin{aligned}
A_{(-a,t)} &= \sum 2\pi y \delta l \\
&= \sum 2\pi y \sqrt{\delta x^2 + \delta y^2} \\
&= 2\pi \sum y \left[1 + \left(\frac{\delta y}{\delta x} \right)^2 \right]^{1/2} \delta x
\end{aligned}$$

In the limit as $\delta x \to 0$ this becomes the required integral.